$\mathbb{A}^1$-homotopical classification of principal G-bundles

MARC HOYOIS

(joint work with Aravind Asok and Matthias Wendt)

Let k be a commutative ring, G a reductive algebraic group over k, and X a smooth affine k-scheme. We are interested in understanding the set of isomorphism classes of generically trivial G-torsors over X. By a theorem of Nisnevich [7], if k is regular, this is equivalently the set $H^1_{\mathrm{Nis}}(X, G)$ of G-torsors that are trivial locally in the Nisnevich topology.

Theorem 1 ([2, 3]). *Let k be an infinite field, G an isotropic reductive k-group, and X a smooth affine k-scheme. Then there is a bijection*

$$H^1_{\mathrm{Nis}}(X, G) \simeq [X, BG]_{\mathbb{A}^1},$$

where the right-hand side denotes the set of maps in the $\mathbb{A}^1$-homotopy category over k [6].

If G is GL_n, SL_n, or Sp_{2n}, the above result holds for k any commutative ring which admits a regular ring homomorphism from a Dedekind domain with perfect residue fields.

The usefulness of Theorem 1 stems from the fact that the right-hand side is more amenable to computation, using tools from ($\mathbb{A}^1$-)homotopy theory.

The prototypical case of Theorem 1, when $G = GL_n$ and k is a perfect field, was established by Morel [5]. This was extended to $G = SL_n$ by Asok and Fasel [1], and a simplified proof applying also to $G = Sp_{2n}$ was later found by Schlichting [9], still under the assumption that k is a perfect field. Our approach is completely independent of Morel's and allows us to remove all assumptions on k, except the (obviously necessary) assumption that $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes. In other words, our proof of Theorem 1 proceeds in two independent steps:

Theorem 2. *Theorem 1 holds for any commutative ring k and k-group scheme G such that $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes.*

Theorem 3. *If k is an infinite field and G is an isotropic reductive k-group, then $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes.*

The second part of Theorem 1 follows from Theorem 2 and the partial solution of the Bass–Quillen conjecture by Lindel and Popescu [8].

The proof of Theorem 3 is a variant of arguments of Colliot-Thélène and Ojanguren [4], combined with an analog of Quillen's patching theorem for G-torsors.

The proof of Theorem 2 relies on a new characterization of the Nisnevich topology. Recall that a Nisnevich cover is an étale cover that is surjective on k-points for every field k.

Theorem 4. *The Nisnevich topology on the category of schemes is generated by the following types of covers:*

(1) *open covers;*

(2) $\{\operatorname{Spec} B \to \operatorname{Spec} A, \operatorname{Spec} A[1/f] \hookrightarrow \operatorname{Spec} A\}$, *where* $A \to B$ *is an étale ring homomorphism inducing an isomorphism* $A/fA \cong B/fB$.

On the category of affine schemes, covers of type (2) suffice.

If in (2) we replace $\operatorname{Spec} A[1/f] \hookrightarrow \operatorname{Spec} A$ by an arbitrary open immersion $U \hookrightarrow \operatorname{Spec} A$, requiring $\operatorname{Spec} B \to \operatorname{Spec} A$ to be an isomorphism over the closed complement of U, then the result is well known and goes back to Morel and Voevodsky [6]. Thus, the main innovation of Theorem 4 is that it suffices to consider complements of hypersurfaces, which leads to a simple set of generators for the Nisnevich topology on affine schemes.

References

[1] A. Asok, J. Fasel, *Algebraic vector bundles on spheres*, J. Topology **7** (2014), no. 3, pp. 894–926

[2] A. Asok, M. Hoyois, and M. Wendt, *Affine representability results in* $\mathbb{A}^1$*-homotopy theory I: vector bundles*, 2015, `http://arxiv.org/abs/1506.07093`

[3] A. Asok, M. Hoyois, and M. Wendt, *Affine representability results in* $\mathbb{A}^1$*-homotopy theory II: principal bundles and homogeneous spaces*, 2015, `http://arxiv.org/abs/1507.08020`

[4] J.-L. Colliot-Thélène, M. Ojanguren, *Espaces principaux homogènes localement triviaux*, Publ. Math. I.H.É.S. **75** (1992), pp. 97–122

[5] F. Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, volume 2052, Springer, Heidelberg, 2012

[6] F. Morel and V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Publ. Math. I.H.É.S. **90** (1999), pp. 45–143

[7] Y. A. Nisnevich, *Éspaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind*, C. R. Acad. Sci. Paris Sér. I Math., **299** (1984), pp. 5–8

[8] D. Popescu, *Polynomial rings and their projective modules*, Nagoya Math. J. **113** (1989), pp. 121–128

[9] M. Schlichting, *Euler class groups, and the homology of elementary and special linear groups*, 2016, `http://arxiv.org/abs/1502.05424`