1 Nisnevich topology

Let $U \to X$ be a morphism of schemes and let $x \in X$. Call $U \to X$ *completely decomposed at* x if there exists $u \in U$ lying above x such that the residual field extension $k(x) \to k(u)$ is an isomorphism. Such points u are in bijection with lifts in the diagram

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] \arrow[ur, dashed] & X, \end{tikzcd} \tag{1}$$

the lift $\operatorname{Spec} k(x) \to U$ corresponding to u being induced by $\mathcal{O}_{U,u} \to k(u) \cong k(x)$. From this we deduce that for a cartesian square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

with $f(y) = x$, if $U \to X$ is completely decomposed at x, then $V \to Y$ is completely decomposed at y.

A family of morphisms of schemes $\{f_i \colon U_i \to X\}_{i \in I}$ is called a *Nisnevich covering* if I is finite, each morphism f_i is étale of finite type, and for every $x \in X$ there exists $i \in I$ such that f_i is completely decomposed at x. Any Zariski covering is thus a Nisnevich covering, while any Nisnevich covering is an étale covering. It is clear that a finite family $\{f_i \colon U_i \to X\}_{i \in I}$ is a Nisnevich covering if and only if the single induced morphism $\coprod_{i \in I} U_i \to X$ is a Nisnevich covering (considered as a singleton family). Using the fact about pullbacks established above, we get:

Proposition 1.1. *Let S be a scheme and $\mathcal{C}$ a full subcategory of $\mathcal{S}\mathrm{ch}/S$ such that the pullback in $\mathcal{S}\mathrm{ch}/S$ of a diagram*

$$\begin{tikzcd} & U \arrow[d, "p"] \\ Y \arrow[r] & X \end{tikzcd}$$

in $\mathcal{C}$ in which p is étale of finite type is in $\mathcal{C}$. Then Nisnevich coverings form a basis for a topology on $\mathcal{C}$.

The induced topology on $\mathcal{C}$ is called the *Nisnevich topology*. It is finer than the Zariski topology on $\mathcal{C}$ but coarser than the étale one. In particular, the Nisnevich topology is subcanonical. We shall use the notations $\mathcal{C}_{\mathrm{Zar}}$, $\mathcal{C}_{\mathrm{Nis}}$, and $\mathcal{C}_{\text{ét}}$ to denote the sites whose underlying category is $\mathcal{C}$ and whose topology is respectively the Zariski topology, the Nisnevich topology, and the étale topology. The identity functor on $\mathcal{C}$ is thus a morphism of sites $\mathcal{C}_{\text{ét}} \to \mathcal{C}_{\mathrm{Nis}}$ as well as a morphism of sites $\mathcal{C}_{\mathrm{Nis}} \to \mathcal{C}_{\mathrm{Zar}}$. These are in fact ringed sites by faithfully flat descent.

If X is any scheme over S, we shall always make the abuse of denoting by X the functor $\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/S}(?, X)$ on $\mathcal{C}^\circ$, considered as an object of $\mathcal{S}\mathrm{hv}(\mathcal{C}_{\mathrm{Nis}})$ (the category $\mathcal{C}$ itself being fixed by the context). Note that X need not be an object of $\mathcal{C}$ for this functor to be defined and to be a sheaf. Given a monomorphism of schemes $Y \hookrightarrow X$, the notation X/Y is always used in that sense: it denotes the (pointed) sheaf associated to the presheaf $U \mapsto \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/S}(U, X) / \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/S}(U, Y)$.

Proposition 1.2. *Let $U \to X$ be an étale morphism and let $x \in X$. Then the following are equivalent:*

1. *$U \to X$ is completely decomposed at x;*

2. *the morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ has a section.*

Proof. 1 $\Rightarrow$ 2. The morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ is again étale and completely decomposed at the closed point of $\operatorname{Spec} \mathcal{O}^h_{X,x}$. It follows from [**Mil80**, Theorem 4.2(d)] that it has a section.

2 $\Rightarrow$ 1. By extending the scalars along $\mathcal{O}^h_{X,x} \to k(x)$, we see that the map $U \times_X \operatorname{Spec} k(x) \to \operatorname{Spec} k(x)$ has a section. This is clearly equivalent to the existence of a lift in (1). $\square$

A cartesian square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd} \tag{2}$$

is called an *elementary distinguished square* if i is an open immersion, p is étale, and $p^{-1}(X - i(U)) \to X - i(U)$ is an isomorphism for the reduced structures (or equivalently, for some closed subscheme structures).

We will often use the fact that the square

$$\begin{tikzcd} V \arrow[d, "p"'] & p^{-1}(Z) \arrow[l] \arrow[d] \\ X & Z \arrow[l] \end{tikzcd}$$

is cartesian whenever p is étale, $Z \subseteq X$ is a closed subset, and both Z and $p^{-1}(Z)$ are endowed with the reduced structures. Indeed, the pullback $V \times_X Z \to V$ is a closed immersion whose underlying closed subset is $p^{-1}(Z)$, and $V \times_X Z$ is also reduced since an étale scheme over a reduced scheme is reduced (ref). Thus, by the uniqueness of reduced subschemes on a given closed subset, the map $p^{-1}(Z) \to V \times_X Z$ induced by this square is an isomorphism.

Proposition 1.3. *Let S be a scheme and let $\mathcal{C}$ be a full subcategory of $\mathcal{S}\mathrm{ch}/S$ as in Proposition 1.1. Then elementary distinguished squares in $\mathcal{C}$ are cocartesian in $\mathcal{S}\mathrm{hv}(\mathcal{C}_{\mathrm{Nis}})$.*

Proof. Consider the elementary distinguished square (2), and let $Z = X - i(U)$, $Z' = p^{-1}(Z)$. Let F be a sheaf and let $u\colon U \to F$ and $v\colon V \to F$ be maps agreeing on $U \times_X V$. We have to show that there is a unique map $X \to F$ through which both u and v factor. Since F is a sheaf and $\{U \to X, V \to X\}$ is a Nisnevich covering of X, maps $X \to F$ are in bijection with matching pairs of maps $U \to F$ and $V \to F$, i.e., maps that agree on the four intersections $U \times_X U$, $U \times_X V$, $V \times_X U$, and $V \times_X V$. Any map $X \to F$ having the desired property will correspond through this bijection to u and v, so it is necessary and sufficient to show that u and v form a matching family. They clearly match on $U \times_X V$ and $V \times_X U$ by hypothesis. Also obvious is that u matches with itself on $U \times_X U$, because the two projections $U \times_X U \rightrightarrows U$ coincide (the immersion $U \hookrightarrow X$ being a monomorphism in $\mathcal{C}$).

It remains to show that $vp_1 = vp_2$ where p_1 and p_2 are the two projections $V \times_X V \rightrightarrows V$. Consider the diagonal morphism $\Delta\colon V \to V \times_X V$ and the projection $\pi\colon U \times_X V \times_X V \to V \times_X V$; Δ is an open immersion because p is unramified, and π is also an open immersion as a pullback of i. We claim that these two maps form a covering of $V \times_X V$. Since the maps $U \hookrightarrow X$ and $Z \hookrightarrow X$ cover X, their pullbacks along $V \times_X V \to X$ cover $V \times_X V$. It will thus suffice to show that the pullback of $Z \hookrightarrow X$ factors through Δ. In fact, the square

$$\begin{tikzcd} V \times_X V \arrow[d] & Z' \arrow[l] \arrow[d, "\cong"] \\ X & Z, \arrow[l] \end{tikzcd}$$

in which the top arrow is the composition of the closed immersion $Z' \to V$ and the diagonal Δ, is seen to be cartesian by direct inspection.

Thus, since F is a sheaf, to prove that $vp_1 = vp_2$, it is sufficient to prove that $vp_1\Delta = vp_2\Delta$ and that $vp_1\pi = vp_2\pi$. The former is obvious and the latter is seen to hold by examination of the contour of the commutative diagram

$$\begin{tikzcd} U \times_X V \times_X V \arrow[r, "\pi"] \arrow[d, shift left] \arrow[d, shift right] & V \times_X V \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "v"] \\ U \arrow[r, "u"'] & F. \end{tikzcd}$$

□

Corollary 1.4. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{S}\mathrm{ch}/S$ as in Proposition 1.1. If (2) is an elementary distinguished square in $\mathcal{C}$ and if F is a sheaf of abelian groups on $\mathcal{C}_{\mathrm{Nis}}$, then there is a long exact sequence*

$$\cdots \to H^i_{\mathrm{Nis}}(X, F) \to H^i_{\mathrm{Nis}}(U, F) \oplus H^i_{\mathrm{Nis}}(V, F) \to H^i_{\mathrm{Nis}}(U \times_X V, F) \to H^{i+1}_{\mathrm{Nis}}(X, F) \to \cdots$$

in which all maps are natural in F.

Proof. For $X \in \mathcal{C}$, we let $\mathbf{Z}[X]$ denote the free abelian sheaf on $\mathrm{Hom}_{\mathcal{C}}(?, X)$. Then $H^i_{\mathrm{Nis}}(X, ?) = \mathrm{Ext}^i(\mathbf{Z}[X], ?)$, so the long exact sequence comes from the short exact sequence

$$0 \to \mathbf{Z}[U \times_X V] \to \mathbf{Z}[U] \oplus \mathbf{Z}[V] \to \mathbf{Z}[X] \to 0.$$

The exactness in the middle and on the right is a consequence of Proposition 1.3, while exactness on the left follows from the fact that $U \times_X V \to V$ is a monomorphism. □

Lemma 1.5. *Let $p\colon U \to X$ be an étale morphism between Noetherian schemes. If p is completely decomposed at every generic point of X, then $U \to X$ has a rational section, i.e., there exists a dense open subset $X' \subseteq X$ such that $p\colon p^{-1}(X') \to X'$ has a section.*

Proof. To give a rational section $X \to U$ is equivalent to giving a rational section $X_i \to U$ for each irreducible component X_i of X ([**EGA1**, 7.1.7]), so we may assume that X is irreducible. Let x be its generic point. Since U is Noetherian, p is of finite type. It follows that rational sections of p are in bijection with pairs (u, s) where $u \in U$ lies above x and $s\colon \mathcal{O}_{U,u} \to \mathcal{O}_{X,x}$ is a local section of $p_u\colon \mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ ([**EGA1**, 7.1.13]). Take u such that $k(x) \to k(u)$ is an isomorphism. Because p is étale, $\mathcal{O}_{U,u}$ is essentially étale over $\mathcal{O}_{X,x}$ ([**EGA4**, 18.4.12]), and therefore there exists a local lift in the diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & \mathcal{O}^h_{X,x} \\ \mathcal{O}_{U,u} \arrow[ur, dashed] & \end{tikzcd}$$

([**EGA4**, 18.6.2]). But $\mathcal{O}_{X,x}$ is an Artinian local ring ([**EGA1**, 7.1.5]) and in particular is complete, so $\mathcal{O}_{X,x} \to \mathcal{O}^h_{X,x}$ is an isomorphism. This gives the desired section. □

Proposition 1.6. *Let $p\colon U \to X$ be a Nisnevich covering with U and X Noetherian. Then there exists a sequence*

$$\varnothing = Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_0 = X$$

of closed subsets of X such that, for each i, $p^{-1}(Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ has a section (where for $i \geq 1$, Z_i and $p^{-1}(Z_i)$ are endowed with the reduced structures).

Proof. By Lemma 1.5, there exists a dense open subset $X' \subseteq X$ such that $p\colon p^{-1}(X') \to X'$ has a section, so we can let $Z_1 = X - X'$. Then $p^{-1}(Z_1) \to Z_1$ is again a Nisnevich covering since it is the pullback of such a covering, so we can repeat the construction with $p^{-1}(Z_1) \to Z_1$ instead of $U \to X$. We thus construct a descending sequence $X = Z_0, Z_1, Z_2, \ldots$ of closed subsets of X with the desired property and with $Z_i \neq Z_{i+1}$ unless $Z_i = \varnothing$. But the sequence must stabilize since X is Noetherian, so $Z_n = \varnothing$ for some n. □

Theorem 1.7. *Let S be a scheme and let $\mathcal{C}$ be a full subcategory of $\mathcal{S}\mathrm{ch}/S$ as in Proposition 1.1 and moreover consisting of Noetherian schemes. Let $\mathcal{D}$ be any category. Then a functor $F\colon \mathcal{C}^\circ \to \mathcal{D}$ is a sheaf for the Nisnevich topology on $\mathcal{C}$ if and only if the following conditions hold:*

1. *$F(\varnothing)$ is terminal in $\mathcal{D}$;*
2. *for any $X \in \mathcal{C}$ and any elementary distinguished square (2), the induced square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \to & F(U \times_X V) \end{array}$$

is cartesian in $\mathcal{D}$.

Proof. Since F is a sheaf if and only if $\mathrm{Hom}_{\mathcal{D}}(d, ?) \circ F$ is a sheaf of sets for every $d \in \mathcal{D}$, and since the two conditions hold of F if and only if they hold of $\mathrm{Hom}_{\mathcal{D}}(d, ?) \circ F$ for every $d \in \mathcal{D}$, we can assume that $\mathcal{D} = \mathcal{S}\mathrm{et}$. Propostion 1.3 and the Yoneda lemma take care of the "only if" direction. Conversely, assume that the presheaf F satisfies the two conditions. If $U, V \in \mathcal{C}$, then

$$\begin{array}{ccc} \varnothing & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \to & U \amalg V \end{array}$$

is an elementary distinguished square, and using conditions 1 and 2 we deduce that $F(U \amalg V) \to F(U) \times F(V)$ is an isomorphism. By induction, we get that the map

$$F(U_1 \amalg \cdots \amalg U_n) \to F(U_1) \times \cdots \times F(U_n)$$

is an isomorphism for any schemes $U_1, \ldots, U_n$ in $\mathcal{C}$. Thus, F will satisfy the sheaf condition for an arbitrary Nisnevich covering $\{U_i \to X\}_i$ if and only if it does for the Nisnevich covering $\coprod_i U_i \to X$ consisting of a single map.

To any Nisnevich covering $p\colon U \to X$ in $\mathcal{C}$ we associate the smallest integer $n \geq 0$ such that there exists a sequence

$$\varnothing = Z_{n+1} \subseteq Z_n \subseteq \cdots \subseteq Z_0 = X$$

of closed subsets of X as in Proposition 1.6. We shall prove that the sequence

$$F(X) \to F(U) \rightrightarrows F(U \times_X U) \tag{3}$$

is exact by induction on n. If $n = 0$, then p has a section, and therefore (3) is split exact. Suppose $n \geq 1$, and let $X' = X - Z_n$. Then $X' \times_X U \to X'$ is a Nisnevich covering whose associated integer is at most $n - 1$. By induction hypothesis, the sequence

$$F(X') \to F(X' \times_X U) \rightrightarrows F(X' \times_X U \times_X U) \tag{4}$$

is exact. Choose a section $s\colon Z_n \to p^{-1}(Z_n)$ of p. By [**EGA4**, 17.9.3], s is an open immersion. Let $V = s(Z_n) \cup p^{-1}(X')$; this is an open subset of U. Then the square

$$\begin{array}{ccc} X' \times_X V & \to & V \\ \downarrow & & \downarrow{\scriptstyle p|V} \\ X' & \longrightarrow & X \end{array}$$

is an elementary distinguished square: the morphism $(p|V)^{-1}(Z_n) = s(Z_n) \to Z_n$ is indeed left inverse to $s\colon Z_n \to s(Z_n)$ which is an isomorphism since s is an open immersion. By condition 2, the induced square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X') \\ \downarrow & & \downarrow \\ F(V) & \to & F(X' \times_X V) \end{array} \tag{5}$$

is cartesian.

Let us finally derive the exactness of (3). First, we observe that the square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X') \times F(V) \arrow[d] \\ F(U) \arrow[r] & F(X' \times_X U) \times F(V) \end{tikzcd}$$

is commutative and that, by (4) and (5), the right arrow and the top arrow are injective. Therefore, the left arrow, which is the first arrow in (3), is injective. Now suppose given an element $u \in F(U)$ such that $p_1^*(u) = p_2^*(u)$ in $F(U \times_X U)$. Let $v \in F(V)$ and $w \in F(X' \times_X U)$ be the restrictions of u. Then $p_1^*(w) = p_2^*(w)$, so by exactness of (4), w comes from an element $x' \in F(X')$. Now x' and v have the same image in $F(X' \times_X V)$, so by (5) we obtain an element $x \in F(X)$ which maps to $u \in F(U)$. This concludes the proof of the exactness of (3) and of the theorem. □

2 Thom spaces

In this section S is a scheme and $\mathcal{C}$ is a fixed full subcategory of $\mathcal{S}\mathrm{ch}/S$ as in Proposition 1.1. All schemes considered are schemes over S and morphisms are S-morphisms.

Let X be a scheme and let $\mathcal{E}$ be a quasi-coherent X-module. We denote by $\mathbf{V}(\mathcal{E})$ the associated vector bundle over X defined by $\mathbf{V}(\mathcal{E}) = \operatorname{Spec} S(\mathcal{E})$, where $S(\mathcal{E})$ is the free commutative X-algebra on $\mathcal{E}$, and we denote by $\mathbf{P}(\mathcal{E})$ the associated projective bundle over X, defined by $\mathbf{P}(\mathcal{E}) = \operatorname{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ has the usual grading.

Any morphism $u\colon \mathcal{E} \to \mathcal{F}$ of quasi-coherent X-modules induces a morphism $\mathbf{V}(u)\colon \mathbf{V}(\mathcal{F}) \to \mathbf{V}(\mathcal{E})$ over X making $\mathbf{V}$ into a functor. Moreover, if u is an epimorphism, $\mathbf{V}(u)$ is a closed immersion. In this case, u also induces a closed immersion $\mathbf{P}(u)\colon \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ over X in a functorial way. If $f\colon Y \to X$ is a morphism of schemes and $\mathcal{E}$ is a quasi-coherent X-module, we have pullback squares

$$\begin{tikzcd} \mathbf{V}(f^*(\mathcal{E})) \arrow[r] \arrow[d] & \mathbf{V}(\mathcal{E}) \arrow[d] \\ Y \arrow[r] & X, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{P}(f^*(\mathcal{E})) \arrow[r] \arrow[d] & \mathbf{P}(\mathcal{E}) \arrow[d] \\ Y \arrow[r] & X. \end{tikzcd}$$

We denote by $\mathcal{M}\mathrm{od}$ the category of pairs $(X, \mathcal{E})$ where X is a scheme and $\mathcal{E}$ is a quasi-coherent X-module. A morphism from $(X, \mathcal{E})$ to $(Y, \mathcal{F})$ in this category consists of a morphism of schemes $f\colon X \to Y$ and a morphism of X-modules $f^*(\mathcal{F}) \to \mathcal{E}$. If we only consider epimorphisms $f^*(\mathcal{F}) \to \mathcal{E}$, since functors of the form f^* are left adjoint and hence preserve epimorphisms, we obtain a subcategory $\mathcal{M}\mathrm{od}_{\mathrm{epi}}$ of $\mathcal{M}\mathrm{od}$. From the previous considerations we see that the category $\mathcal{M}\mathrm{od}$ is the domain of the vector bundle functor $\mathbf{V}$, while the projective bundle functor $\mathbf{P}$ is well-defined on $\mathcal{M}\mathrm{od}_{\mathrm{epi}}$.

Proposition 2.1. *Let X be a scheme and let $\mathcal{E}$ be a locally free X-module of finite rank. Then $p\colon \mathbf{V}(\mathcal{E}) \to X$ is a strict $\mathbf{A}^1_X$-homotopy equivalence over X.*

Proof. Let $i\colon X \to \mathbf{V}(\mathcal{E})$ be the zero section of p, i.e., the closed immersion induced by $\mathcal{E} \to 0$. Choose an open covering $\{U_i\}_i$ of X and trivializations $p^{-1}(U_i) \cong U_i \times \mathbf{A}^{n_i}_{\mathbf{Z}}$ over U_i, and define $h_i\colon p^{-1}(U_i) \times \mathbf{A}^1_{\mathbf{Z}} \to p^{-1}(U_i)$ to be the composition

$$p^{-1}(U_i) \times \mathbf{A}^1_{\mathbf{Z}} \xrightarrow{\cong} U_i \times \mathbf{A}^{n_i}_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} \xrightarrow{\mathrm{id}\times\mu} U_i \times \mathbf{A}^{n_i}_{\mathbf{Z}} \xrightarrow{\cong} p^{-1}(U_i),$$

where μ corresponds to the map

$$\mathbf{Z}[T_1, \ldots, T_{n_i}] \to \mathbf{Z}[T_1, \ldots, T_{n_i}] \otimes \mathbf{Z}[T], \quad T_i \mapsto T_i \otimes T.$$

Because this map commutes with any automorphism of $\mathbf{Z}[T_1, \dots, T_{n_i}]$, the maps h_i and h_j agree over $U_i \cap U_j$. The morphisms h_i can thus be glued to give a morphism $h\colon \mathbf{V}(\mathcal{E}) \times \mathbf{A}^1_{\mathbf{Z}} = \mathbf{V}(\mathcal{E}) \times_X \mathbf{A}^1_X \to \mathbf{V}(\mathcal{E})$ over X which is clearly a homotopy from ip to the identity. □

It will be convenient to introduce the category $\mathcal{I}\mathrm{mm}$ of closed immersions. An object in $\mathcal{I}\mathrm{mm}$ is a closed immersion $i\colon Z \hookrightarrow X$, and a morphism from $i'\colon Z' \hookrightarrow X'$ to $i\colon Z \hookrightarrow X$ is a cartesian square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "u"'] & X' \arrow[d, "v"] \\ Z \arrow[r, "i"'] & X, \end{tikzcd}$$

also written $(u, v)\colon i' \to i$. A closely related category, denoted $\mathcal{I}\mathrm{d}$, has as objects the pairs $(X, \mathcal{I})$ where X is a scheme and $\mathcal{I}$ is a quasi-coherent X-ideal, and as set of morphism from $(X', \mathcal{I}')$ to $(X, \mathcal{I})$ the subset of all morphisms $f\colon X' \to X$ satisfying $f^{-1}(\mathcal{I}) = \mathcal{I}'$. Here $f^{-1}(\mathcal{I})$ denotes the quasi-coherent X'-module image of $f^*(\mathcal{I}) \to f^*(\mathcal{O}_X) \cong \mathcal{O}_{X'}$.

Proposition 2.2. *The categories* $\mathcal{I}\mathrm{mm}$ *and* $\mathcal{I}\mathrm{d}$ *are equivalent.*

Proof. Of course, the functor $\mathcal{I}\mathrm{mm} \to \mathcal{I}\mathrm{d}$ sends $i\colon Z \hookrightarrow X$ to the kernel of $\mathcal{O}_X \to i_*(\mathcal{O}_Z)$, and given the equivalence between closed subschemes of a given scheme X and its quasi-coherent X-ideals, it remains to observe that if

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "u"'] & X' \arrow[d, "v"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

is cartesian, then $v^{-1}(\mathcal{I}) = \mathcal{I}'$. □

Using the fact that closed immersions are monomorphisms in $\mathcal{S}\mathrm{ch}/S$, we deduce categorically the following result.

Proposition 2.3. *The functor* $\mathcal{I}\mathrm{mm} \to (\mathcal{S}\mathrm{ch}/S)^2$ *sending* $i\colon Z \hookrightarrow X$ *to* (Z, X) *creates pullbacks. In particular, pullbacks exists in* $\mathcal{I}\mathrm{mm}$.

Note however that in general products do not exist in $\mathcal{I}\mathrm{mm}$, since it has no final object.

A morphism $(u, v)\colon i' \to i$ in $\mathcal{I}\mathrm{mm}$ is called *flat* (resp. *étale*) if v (and hence u) is flat (resp. étale) and it is called *Nisnevich* if it is étale and moreover u is an isomorphism; we also say that i' is flat, étale, or Nisnevich over i. The significance of the latter definition is of course that the morphisms $v\colon X' \to X$ and $X - i(Z) \hookrightarrow X$ form an elementary distinguished square if (u, v) is Nisnevich.

Lemma 2.4. *Let*

$$\begin{tikzcd} Z_1 \arrow[r, hook, "i_1"] \arrow[d, "u_1"'] & X_1 \arrow[d, "v_1"] \\ Z \arrow[r, hook, "i"] & X, \end{tikzcd} \qquad \begin{tikzcd} Z_2 \arrow[r, hook, "i_2"] \arrow[d, "u_2"'] & X_2 \arrow[d, "v_2"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

be étale morphisms in $\mathcal{I}\mathrm{mm}$. *If* $u_1 = u_2$, *then* i_1 *and* i_2 *are related by a zigzag of Nisnevich morphisms.*

Proof. Let $u = u_1 = u_2$ and $W = Z_1 = Z_2$. We have a cartesian square

$$\begin{tikzcd} W \times_Z W \arrow[r] \arrow[d, "i_1 \times_i i_2"'] & Z \arrow[d, "i"] \\ X_1 \times_X X_2 \arrow[r] & X. \end{tikzcd}$$

Since u is unramified, the diagonal $\Delta_u\colon W \hookrightarrow W \times_Z W$ is an open immersion. Let Y be the closed subset of $X_1 \times_X X_2$ image of the complement of $\Delta_u(W)$ in $W \times_Z W$, and let j be the closed immersion $W \hookrightarrow X_1 \times_X X_2 - Y$. Then direct inspection shows that the squares

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, equal] & X_1 \times_X X_2 - Y \arrow[d] \\ W \arrow[r, "i_\nu"'] & X_\nu \end{tikzcd}$$

are cartesian for $\nu = 1, 2$, and since v_1 and v_2 are étale they define Nisnevich morphisms $j \to i_1$ and $j \to i_2$. □

We define a functor $Q\colon \mathcal{I}\mathrm{mm} \to \mathcal{S}\mathrm{hv}_\bullet(\mathcal{C}_{\mathrm{Nis}})$ by $Q(i\colon Z \hookrightarrow X) = X/(X - i(Z))$. For a cartesian square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X, \end{tikzcd}$$

we have $f(X' - i'(Z')) \subset X - i(Z)$, whence the map of pointed sheaves $Q(i') \to Q(i)$.

Proposition 2.5. *Let $(f, u)\colon (Y, \mathcal{F}) \to (X, \mathcal{E})$ be a morphism in $\mathcal{M}\mathrm{od}_{\mathrm{epi}}$. Then the square*

$$\begin{tikzcd} Y \arrow[r, hook, "i_Y"] \arrow[d, "f"'] & \mathbf{V}(\mathcal{F}) \arrow[d, "\mathbf{V}(f,u)"] \\ X \arrow[r, hook, "i_X"] & \mathbf{V}(\mathcal{E}), \end{tikzcd}$$

in which the horizontal maps are the zero sections, is cartesian.

Proof. This square is obviously commutative. By Proposition 2.1, it suffices to show that $\mathbf{V}(f, u)^{-1}(\mathcal{I}_X) = \mathcal{I}_Y$, where $\mathcal{I}_X$ (resp. $\mathcal{I}_Y$) is the ideal of i_X (resp. of i_Y). This question is local on X and Y, so we can assume that $f = \operatorname{Spec}\phi$ where $\phi\colon A \to B$ is a morphism of rings and u corresponds to a surjective map $E \otimes_A B \to F$ of B-modules. Then $\mathbf{V}(f, u)$ is associated to the obvious map $S_A(E) \to S_B(F)$ and $\mathcal{I}_X$ corresponds the ideal $S_A(E)_+$ generated by elements of positive degree, and similarly $\mathcal{I}_Y$ corresponds to $S_B(F)_+$. Then it is clear that the extension of $S_A(E)_+$ in $S_B(F)$ is $S_B(F)_+$, as was to be shown. □

This proposition shows that we can consider $\mathbf{V}$ as a functor $\mathcal{M}\mathrm{od}_{\mathrm{epi}} \to \mathcal{I}\mathrm{mm}$ sending $(X, \mathcal{E})$ to the zero section $X \hookrightarrow \mathbf{V}(\mathcal{E})$. The *Thom space* functor Th is then the composition

$$\mathcal{M}\mathrm{od}_{\mathrm{epi}} \xrightarrow{\mathbf{V}} \mathcal{I}\mathrm{mm} \xrightarrow{Q} \mathcal{S}\mathrm{hv}_\bullet(\mathcal{C}_{\mathrm{Nis}}).$$

That is, if X is a scheme over S and $\mathcal{E}$ is a quasi-coherent X-module, then

$$\mathrm{Th}(\mathcal{E}) = \mathrm{Th}(X, \mathcal{E}) = \mathbf{V}(\mathcal{E})/(\mathbf{V}(\mathcal{E}) - i_{\mathcal{E}}(X)),$$

where $i_{\mathcal{E}}\colon X \hookrightarrow \mathbf{V}(\mathcal{E})$ is the zero section.

If $\mathcal{E}$ is a quasi-coherent X-module, the projection $\mathcal{E} \oplus \mathcal{O}_X \to \mathcal{O}_X$ induces a canonical closed immersion $X \cong \mathbf{P}(\mathcal{O}_X) \hookrightarrow \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)$.

Proposition 2.6. *Let $X \in \mathcal{C}$ and let $\mathcal{E}$ be a quasi-coherent X-module. Then there is an isomorphism*

$$\mathrm{Th}(\mathcal{E}) \cong \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)/(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X) - X)$$

of pointed sheaves on $\mathcal{C}_{\mathrm{Nis}}$ which is natural in $(X, \mathcal{E}) \in \mathcal{M}\mathrm{od}_{\mathrm{epi}}$.

Proof. The cartesian square

$$\begin{tikzcd}
\mathbf{V}(\mathcal{E}) - X \arrow[r] \arrow[d] & \mathbf{V}(\mathcal{E}) \arrow[d] \\
\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X) - X \arrow[r] & \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)
\end{tikzcd}$$

is an elementary distinguished square (all maps being open immersions), so by Proposition 1.3, it is a cocartesian square in $\mathcal{S}\mathrm{hv}(\mathcal{C}_{\mathrm{Nis}})$. The lemma follows. ☐

We recall some features and establish some notation about the blowup construction. Let $\mathcal{I}\mathrm{d}_{\mathrm{ft}}$ (resp. $\mathcal{I}\mathrm{d}_{\mathrm{inv}}$) be the subcategory of $\mathcal{I}\mathrm{d}$ spanned by pairs $(X, \mathcal{I})$ in which $\mathcal{I}$ is an X-module of finite type (resp. is an invertible X-module). Under the equivalence of Proposition 2.2, $\mathcal{I}\mathrm{d}_{\mathrm{ft}}$ corresponds to the full subcategory $\mathcal{I}\mathrm{mm}_{\mathrm{lfp}}$ of $\mathcal{I}\mathrm{mm}$ consisting of closed immersions locally of finite presentation. The blowup construction provides a right adjoint to the inclusion $\mathcal{I}\mathrm{d}_{\mathrm{inv}} \hookrightarrow \mathcal{I}\mathrm{d}_{\mathrm{ft}}$. Let X be a scheme and let $\mathcal{I}$ be a quasi-coherent X-ideal of finite type. Let $\mathcal{S}_{\mathcal{I}}$ denote the quasi-coherent X-algebra $\bigoplus_{d\geq 0} \mathcal{I}^d$. Then we set $B(X, \mathcal{I}) = B(\mathcal{I}) = \mathrm{Proj}(\mathcal{S}_{\mathcal{I}})$. It comes with a canonical map $p_{X,\mathcal{I}} \colon B(X, \mathcal{I}) \to X$ such that the ideal $p^{-1}_{X,\mathcal{I}}(\mathcal{I})$ is the invertible module $\mathcal{O}_{B(X,\mathcal{I})}(1)$, and which happens to be universal among morphisms f to X with the property that $f^{-1}(\mathcal{I})$ is invertible. It is useful to have an explicit construction of the morphism $B(f)$ for $f \colon (X', \mathcal{I}') \to (X, \mathcal{I})$. In general there is a cartesian square

$$\begin{tikzcd}
\mathrm{Proj}(f^*(\mathcal{S}_{\mathcal{I}})) \arrow[r] \arrow[d] & \mathrm{Proj}(\mathcal{S}_{\mathcal{I}}) \arrow[d] \\
X' \arrow[r, "f"'] & X,
\end{tikzcd}$$

and the obvious epimorphism $f^*(\mathcal{S}_{\mathcal{I}}) \to \mathcal{S}_{\mathcal{I}'}$ (which is an isomorphism if f is flat) induces a closed immersion $B(X', \mathcal{I}') \hookrightarrow \mathrm{Proj}(f^*(\mathcal{S}_{\mathcal{I}}))$ over X', whence a morphism $B(X', \mathcal{I}') \to B(X, \mathcal{I})$ over f, which necessarily equals $B(f)$. From this we see that $B(f)$ is a (closed) immersion if f is a (closed) immersion.

Proposition 2.7. *The inclusion* $\mathcal{I}\mathrm{d}_{\mathrm{inv}} \hookrightarrow \mathcal{I}\mathrm{d}$ *creates pullbacks along flat morphisms.*

Proof. This is clear since for f flat, $f^{-1}(\mathcal{I}) = f^*(\mathcal{I})$ which is invertible if $\mathcal{I}$ is invertible. ☐

Let $i \colon Z \hookrightarrow X$ be a closed immersion with X-ideal $\mathcal{I}$ and let $Z_{\mathcal{I}}$ be the closed subscheme of $B(X, \mathcal{I})$ defined by the ideal $p^{-1}_{X,\mathcal{I}}(\mathcal{I})$. Then from Proposition 2.2 we know that $Z_{\mathcal{I}} \to Z$ is the pullback of $p_{X,\mathcal{I}}$ along i, so that $Z_{\mathcal{I}}$ can be identified as a Z-scheme with $\mathrm{Proj}(i^*(\mathcal{S}_{\mathcal{I}}))$. Recall that the conormal sheaf $\mathcal{N}_i$ of i is defined to be the Z-module $i^*(\mathcal{S}_{\mathcal{I}})_1 = i^*(\mathcal{I})$. Let

$$\begin{tikzcd}
Z' \arrow[r, "i'"] \arrow[d, "u"'] & X' \arrow[d, "v"] \\
Z \arrow[r, "i"'] & X,
\end{tikzcd}$$

be a commutative square where i and i' are immersions with ideals $\mathcal{I}$ and $\mathcal{I}'$. Then we have morphisms of Z'-modules

$$u^*(\mathcal{N}_i) = u^* i^*(\mathcal{I}) = (i')^* v^*(\mathcal{I}) \to (i')^*(v^{-1}(\mathcal{I})) \to (i')^*(\mathcal{I}') = \mathcal{N}_{i'},$$

the first one induced by the epimorphism $v^*(\mathcal{I}) \to v^{-1}(\mathcal{I})$ and the second one by the inclusion $v^{-1}(\mathcal{I}) \subset \mathcal{I}'$. This gives a morphism $(Z', \mathcal{N}_{i'}) \to (Z, \mathcal{N}_i)$ in $\mathcal{M}\mathrm{od}$. If our original square was cartesian, then, since $v^{-1}(\mathcal{I}) = \mathcal{I}'$, we obtain a morphism in $\mathcal{M}\mathrm{od}_{\mathrm{epi}}$. Thus we view the conormal sheaf as a functor

$$\mathcal{N} \colon \mathcal{I}\mathrm{mm} \to \mathcal{M}\mathrm{od}_{\mathrm{epi}}.$$

Let $\mathcal{I}\mathrm{mm}_{\mathrm{fl}}$ be the subcategory of $\mathcal{I}\mathrm{mm}_{\mathrm{lfp}}$ with the same objects but in which a morphism is a flat morphism.

Theorem 2.8. *Let $i\colon Z \hookrightarrow X$ be a closed immersion and let $\mathcal{N}_i$ be the conormal sheaf of i. Assume that S is Noetherian and that X and Z are smooth and of finite type over S. Then there is an isomorphism*

$$\mathrm{Th}(\mathcal{N}_i) \cong X/(X - i(Z))$$

in the pointed homotopy category $\mathcal{H}_\bullet(\mathcal{C}_{\mathrm{Nis}})$ which is natural for $i \in \mathcal{I}\mathrm{mm}_{\mathrm{fl}}$.

Before turning to the proof we explain the claim of naturality. We shall in fact construct a zigzag of natural transformations

$$\mathrm{Th} \circ \mathcal{N} \xleftarrow{\alpha} P_1 \xleftarrow{\beta} P_2 \xrightarrow{\gamma} \tilde{Q} \xleftarrow{\delta} Q \tag{6}$$

between functors $\mathcal{I}\mathrm{mm}_{\mathrm{fl}} \to \mathcal{S}\mathrm{hv}_\bullet(\mathcal{C}_{\mathrm{Nis}})$, and we shall prove that all are $\mathbf{A}^1$-equivalences when S is Noetherian and X and Z are smooth and of finite type over S.

We first fix some notations that will be in effect throughout the proof. Given a closed immersion $i\colon Z \hookrightarrow X$, we let j be the composition of i and the zero section $i_0^X\colon X \hookrightarrow X \times \mathbf{A}^1$. Denote by $\mathcal{I}$ and $\mathcal{J}$ the ideals of i and j. Let $p_{\mathcal{I}}\colon B(\mathcal{I}) \to X$ and $p_{\mathcal{J}}\colon B(\mathcal{J}) \to X \times \mathbf{A}^1$ be the blowups of X and $X \times \mathbf{A}^1$ with respect to these ideals, and let $Z_{\mathcal{I}}$ (resp. $Z_{\mathcal{J}}$) be the closed subscheme of $B(\mathcal{I})$ (resp. $B(\mathcal{J})$) defined by the ideal $p_{\mathcal{I}}^{-1}(\mathcal{I})$ (resp. $p_{\mathcal{J}}^{-1}(\mathcal{J})$). The square

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, equal] & X \arrow[d, "i_0^X"] \\ Z \arrow[r, "j"'] & X \times \mathbf{A}^1 \end{tikzcd}$$

is obviously cartesian and defines a morphism $i \to j$ in $\mathcal{I}\mathrm{mm}$. In other words, by Proposition 2.2, $(i_0^X)^{-1}(\mathcal{J}) = \mathcal{I}$. Also, for any morphism $f\colon (X', \mathcal{I}') \to (X, \mathcal{I})$ in $\mathcal{I}\mathrm{d}$, $f \times \mathbf{A}^1$ is a morphism $(X' \times \mathbf{A}^1, \mathcal{J}') \to (X \times \mathbf{A}^1, \mathcal{J})$. In other words, the construction $i \mapsto j$ is a functor $\mathcal{I}\mathrm{mm} \to \mathcal{I}\mathrm{mm}$. Moreover, the morphism $i \to j$ is a natural transformation from the identity to this functor.

Construction of P_1. Applying the functor $\mathbf{P} \circ \mathcal{N}$ to the morphism $i \to j$ in $\mathcal{I}\mathrm{mm}$, we get a closed immersion $\mathbf{P}(\mathcal{N}_i) \hookrightarrow \mathbf{P}(\mathcal{N}_j)$. We let $P_1(i) = \mathbf{P}(\mathcal{N}_j)/\mathbf{P}(\mathcal{N}_i)$. Functoriality is defined using the functoriality of $\mathbf{P} \circ \mathcal{N}$ on $\mathcal{I}\mathrm{mm}$.

For future use, we note that $\mathcal{N}_j \cong \mathcal{N}_i \oplus \mathcal{O}_Z$. Indeed, $j^\bullet(\mathcal{J})$ is the preimage of $i^\bullet(\mathcal{I})$ by the evaluation at 0 map $i^\bullet(\mathcal{O}_X)[t] = j^\bullet(\mathcal{O}_{X \times \mathbf{A}^1}) \to i^\bullet(\mathcal{O}_X)$, so that $j^\bullet(\mathcal{J}) = i^\bullet(\mathcal{I}) \oplus (t)$ as ideals of $i^\bullet(\mathcal{O}_X)[t]$. It follows that $\mathcal{N}_j \cong \mathcal{N}_i \oplus i^\bullet(\mathcal{O}_X/\mathcal{I}) \cong \mathcal{N}_i \oplus \mathcal{O}_Z$ as Z-modules, and this identification is easily proved to be natural in i. Furthermore, the epimorphism $\mathcal{N}_j \to \mathcal{N}_i$ corresponds to the projection $\mathcal{N}_i \oplus \mathcal{O}_Z \to \mathcal{N}_i$.

Construction of P_2. By the functoriality of B and Proposition 2.2 we have a cartesian square

$$\begin{tikzcd} Z_{\mathcal{I}} \arrow[r, hook] \arrow[d] & B(\mathcal{I}) \arrow[d] \\ Z_{\mathcal{J}} \arrow[r, hook] & B(\mathcal{J}), \end{tikzcd}$$

and since the right arrow is a closed immersion, so is $Z_{\mathcal{I}} \to Z_{\mathcal{J}}$. We define the functor P_2 by

$$P_2(i) = Z_{\mathcal{J}}/Z_{\mathcal{I}}.$$

If $f\colon (X', \mathcal{I}') \to (X, \mathcal{I})$ is a morphism in $\mathcal{I}\mathrm{d}$, $P_2(f)$ is defined from $f \times \mathbf{A}^1$ in the same way as $Z_{\mathcal{I}} \to Z_{\mathcal{J}}$ was defined from $(X, \mathcal{I}) \to (X \times \mathbf{A}^1, \mathcal{J})$.

Construction of $\tilde{Q}$. The preimage of $\mathcal{J}$ under $i \times \mathbf{A}^1$ is the sheaf of ideals corresponding to the zero section $i_0^Z\colon Z \hookrightarrow Z \times \mathbf{A}^1$; on an affine piece $U \times \mathbf{A}^1 \cong \mathrm{Spec}(A[T])$ of $Z \times \mathbf{A}^1$, this sheaf is associated to the ideal (T) of $A[T]$ which is a free $A[T]$-module of rank 1, so it is an invertible

$Z \times \mathbf{A}^1$-module. By the universal property of the blowup, there exists a unique lift f_i in the diagram

$$\begin{tikzcd} & B(\mathcal{J}) \arrow[d, "p_{\mathcal{J}}"] \\ Z \times \mathbf{A}^1 \arrow[ur, dashed, "f_i"] \arrow[r, hook, "i \times \mathbf{A}^1"'] & X \times \mathbf{A}^1, \end{tikzcd}$$

and it is a closed immersion. We define

$$\tilde{Q}(i) = B(\mathcal{J})/(B(\mathcal{J}) - f_i(Z \times \mathbf{A}^1)).$$

To see that $\tilde{Q}$ is a functor on $\mathfrak{Imm}_{\mathrm{fl}}$, let

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

be a morphism in $\mathfrak{Imm}_{\mathrm{fl}}$. By Proposition 2.3, the square

$$\begin{tikzcd} (Z' \hookrightarrow Z' \times \mathbf{A}^1) \arrow[r] \arrow[d] & (Z' \hookrightarrow X' \times \mathbf{A}^1) \arrow[d] \\ (Z \hookrightarrow Z \times \mathbf{A}^1) \arrow[r] & (Z \hookrightarrow X \times \mathbf{A}^1) \end{tikzcd}$$

is cartesian in $\mathfrak{Imm}_{\mathrm{lfp}}$. Applying the functor B, which is right adjoint, to this square, we obtain the square

$$\begin{tikzcd} Z' \times \mathbf{A}^1 \arrow[r, "f_{i'}"] \arrow[d] & B(j') \arrow[d] \\ Z \times \mathbf{A}^1 \arrow[r, "f_i"'] & B(j) \end{tikzcd} \tag{7}$$

which is cartesian by Proposition 2.7, whence the morphism of sheaves $\tilde{Q}(i) \to \tilde{Q}(i')$.

Proposition 2.9. *The functors P_2, $\tilde{Q}$, and Q commute with sums and pullbacks along flat morphisms.*

Proof. The fact that Q preserves sums and pullbacks reduces to the following easy fact: the functor $(A \subset B) \mapsto B/A$ on the category of pairs of sets $A \subset B$, where a morphism $f\colon B \to B'$ has to satisfy $f^{-1}(A') = A$, to the category of pointed sets preserves sums and pullbacks.

The functor $\tilde{Q}$ is the composition of $i \mapsto f_i$ and Q, the former being a functor $\mathfrak{Imm}_{\mathrm{fl}} \to \mathfrak{Imm}$ by virtue of (7). We claim that $i \mapsto f_i$ preserves sums and pullbacks along flat morphisms. By Proposition 2.3, it suffices to check that the functors $i \mapsto Z \times \mathbf{A}^1$ and $i \mapsto B(j)$ do; this is obvious for the former, and the latter is the composition of $i \mapsto j$, which obviously has the desired property, and $B\colon \mathfrak{Imm}_{\mathrm{lfp}} \to \mathfrak{Imm}$, which preserves pulbacks along flat morphisms by Proposition 2.7. That B preserves sums is clear from the Proj definition.

For P_2, we first note that $i \mapsto Z_{\mathcal{J}}$ is a functor on $\mathfrak{Imm}_{\mathrm{lfp}}$ that preserves pullbacks along flat morphisms, since $Z_{\mathcal{J}}$ is the domain of $B(i)$ when B is viewed as a functor $\mathfrak{Imm}_{\mathrm{lfp}} \to \mathfrak{Imm}$. It also preserves sums by construction. Now for any $i \to i'$ in $\mathfrak{Imm}_{\mathrm{fl}}$ the square

$$\begin{tikzcd} i \arrow[r] \arrow[d] & i' \arrow[d] \\ j \arrow[r] & j' \end{tikzcd}$$

is cartesian, and applying $i \mapsto Z_{\mathcal{I}}$ to this square, we deduce that $i \mapsto (Z_{\mathcal{I}} \hookrightarrow Z_{\mathcal{J}})$ is a well-defined functor $\mathcal{I}\mathrm{mm}_{\mathrm{lfp}} \to \mathcal{I}\mathrm{mm}$, which preserves sums and pullbacks along flat morphisms by Proposition 2.3. By the same set-theoretic property that we used for Q, we deduce that P_2 preserves sums and pullbacks. □

Construction of α. The closed immersions $\mathbf{P}(\mathcal{N}_i) \hookrightarrow \mathbf{P}(\mathcal{N}_i \oplus \mathcal{O}_Z)$ and $Z = \mathbf{P}(\mathcal{O}_Z) \hookrightarrow \mathbf{P}(\mathcal{N}_i \oplus \mathcal{O}_Z)$ induced by the two projections from $\mathcal{N}_i \oplus \mathcal{O}_Z$ have disjoint images (see [**EGA2**, 4.3.6]), so by Proposition 2.6 we obtain a map $\alpha_i \colon P_1(i) = \mathbf{P}(\mathcal{N}_i \oplus \mathcal{O}_Z)/\mathbf{P}(\mathcal{N}_i) \to \mathrm{Th}(\mathcal{N}_i)$ which is natural for $i \in \mathcal{I}\mathrm{mm}$.

Construction of β. The obvious epimorphism of graded Z-algebras $S(\mathcal{N}_i) \to i^*(\mathcal{S}_{\mathcal{I}})$ is clearly natural in $i \in \mathcal{I}\mathrm{mm}$: if $(u, v) \colon i' \to i$ is a morphism in $\mathcal{I}\mathrm{mm}$, we have a commutative square

$$\begin{tikzcd}
S(\mathcal{N}_{i'}) \arrow[r] \arrow[d] & (i')^*(\mathcal{S}_{\mathcal{I}'}) \arrow[d] \\
u^*(S(\mathcal{N}_i)) \arrow[r] & u^*i^*(\mathcal{S}_{\mathcal{I}})
\end{tikzcd}$$

in which the vertical maps are surjective maps between Z'-algebras. In particular, for $i \to j$, we get a commutative square of closed immersions

$$\begin{tikzcd}
Z_{\mathcal{I}} \arrow[r] \arrow[d] & \mathbf{P}(\mathcal{N}_i) \arrow[d] \\
Z_{\mathcal{J}} \arrow[r] & \mathbf{P}(\mathcal{N}_j),
\end{tikzcd}$$

whence the map $\beta_i \colon P_2(i) = Z_{\mathcal{J}}/Z_{\mathcal{I}} \to P_1(i) = \mathbf{P}(\mathcal{N}_j)/\mathbf{P}(\mathcal{N}_i)$, natural for $i \in \mathcal{I}\mathrm{mm}$.

Construction of γ. $Z_{\mathcal{J}}$ is a natural closed subscheme of $B(\mathcal{J})$, so it suffices to prove that the image of $Z_{\mathcal{I}} \hookrightarrow Z_{\mathcal{J}} \hookrightarrow B(\mathcal{J})$ is disjoint from the image of $f_i \colon Z \times \mathbf{A}^1 \to B(\mathcal{J})$. If we consider the situation over an affine open set $\mathrm{Spec}(A[t]) \cong U \times \mathbf{A}^1 \subset X \times \mathbf{A}^1$ and if $\mathcal{I}(U) = I \subset A$, explicit computations reveal that $f_i(i^{-1}(U) \times \mathbf{A}^1)$ is the closed set $V_+(\bigoplus_{d \geq 0}(I)^d)$ of $p_{\mathcal{J}}^{-1}(U \times \mathbf{A}^1) \cong \mathrm{Proj}(\bigoplus_{d \geq 0}(I, t)^d)$, while the image of $Z_{\mathcal{I}}$ is the closed set $V_+(\bigoplus_{d \geq 0}(t)^d)$ which is obviously disjoint from the former.

Construction of δ. We consider the section $i_1^X \colon X \hookrightarrow X \times \mathbf{A}^1$ at $1 \in \mathbf{A}^1$. Since $i_1^X(X)$ is obviously disjoint from $j(Z)$, $(i_1^X)^{-1}(\mathcal{J}) = \mathcal{O}_X$ is invertible. By the universal property of the blowup, we obtain a unique lift g_i in the diagram

$$\begin{tikzcd}
 & B(\mathcal{J}) \arrow[d, "p_{\mathcal{J}}"] \\
X \arrow[ur, dashed, "g_i"] \arrow[r, hook, "j"'] & X \times \mathbf{A}^1,
\end{tikzcd}$$

which is again a closed immersion. As $i^{-1}(\mathcal{O}_X) = \mathcal{O}_Z$, applying the universal property once more we get $f_i i_1^Z = g_i i$, whence $g_i^{-1}(f_i(Z \times \mathbf{A}^1)) = i(Z)$. It follows that g_i induces a monomorphism $\delta_i \colon \beta_i \colon X/(X - i(Z)) \to B(\mathcal{J})/(B(\mathcal{J}) - f_i(Z \times \mathbf{A}^1))$.

Having completely defined (6), we now turn to the proof of Theorem 2.8.

Lemma 2.10. *Let $\mathcal{E}$ be a locally free X-module of finite type. Then the morphism*

$$\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)/\mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X)/(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X) - X) \cong \mathrm{Th}(\mathcal{E})$$

is an $\mathbf{A}^1$-equivalence in $\mathcal{S}\mathrm{hv}_\bullet(\mathcal{C}_{\mathrm{Nis}})$.

Proof. By the left properness, it suffices to prove that the immersion $\mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X) - X$ is an $\mathbf{A}^1$-equivalence. By [**EGA2**, 8.6.4], there is a commutative diagram

$$\begin{tikzcd} \mathbf{P}(\mathcal{E}) \arrow[r, hook] \arrow[d, equal] & \mathbf{P}(\mathcal{E} \oplus \mathcal{O}_X) - X \arrow[d, "\cong"] \\ \mathbf{P}(\mathcal{E}) \arrow[r, hook] & \mathbf{V}(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-1)) \end{tikzcd}$$

where the bottom arrow is the zero section of the tautological line bundle on $\mathbf{P}(\mathcal{E})$. The lemma now follows from Proposition 2.1. ☐

Proposition 2.11. *If X and Z are smooth over S then β_i is an isomorphism of pointed sheaves. If moreover i is locally of finite presentation, then α_i is an $\mathbf{A}^1$-equivalence.*

Proof. By [**EGA4**, 17.12.1], $\mathcal{N}_i$ is then locally free and $S(\mathcal{N}_i) \to i^*(\mathcal{S}_{\mathcal{J}})$ is an isomorphism, and similarly for j. It follows that the closed immersions $Z_{\mathcal{J}} \hookrightarrow \mathbf{P}(\mathcal{N}_i)$ and $Z_{\mathcal{J}} \hookrightarrow \mathbf{P}(\mathcal{N}_j)$ are both isomorphisms, so β_i is an isomorphism. If $i \in \mathcal{I}\mathrm{mm}_{\mathrm{lfp}}$ then $\mathcal{N}_i$ is also of finite type, so by Lemma 2.10 α_i is an $\mathbf{A}^1$-equivalence. ☐

Lemma 2.12. *Let i be the zero section $Z \hookrightarrow \mathbf{A}^n_Z$ for some $n \geq 0$. Then there is a commutative square*

$$\begin{tikzcd} Z_{\mathcal{J}} \arrow[r] \arrow[d, "\cong"'] & B(\mathcal{J}) \arrow[d, "\cong"] \\ \mathbf{P}^n_Z \arrow[r] & \mathbf{V}(\mathcal{O}_{\mathbf{P}^n_Z}(1)) \end{tikzcd} \tag{8}$$

in which the vertical arrows are isomorphisms and the lower arrow is the zero section and such that the composition

$$\mathbf{A}^n_Z \xrightarrow{g_i} B(\mathcal{J}) \xrightarrow{\cong} \mathbf{V}(\mathcal{O}_{\mathbf{P}^n_Z}(1)) \to \mathbf{P}^n_Z \tag{9}$$

is the canonical open immersion $\mathbf{V}(\mathcal{O}^n_Z) \hookrightarrow \mathbf{P}(\mathcal{O}^n_Z \oplus \mathcal{O}_Z)$ and furthermore the square

$$\begin{tikzcd} Z \times \mathbf{A}^1 \arrow[r, "f_i"] \arrow[d] & B(\mathcal{J}) \arrow[d] \\ Z \arrow[r] & \mathbf{P}^n_Z \end{tikzcd} \tag{10}$$

is cartesian, where the lower arrow is the closed immersion $Z \hookrightarrow \mathbf{P}(\mathcal{O}^n_Z \oplus \mathcal{O}_Z)$ considered in Proposition 2.6.

Proof. We assume first that Z is affine, $Z = \operatorname{Spec} A$, and we omit the subscript Z from the notations. Let $Y = B(\mathbf{A}^{n+1}, \mathcal{J}) = \operatorname{Proj} S$, where S is the graded $A[t_1, \ldots, t_{n+1}]$-algebra $\bigoplus_{d \geq 0}(t_1, \ldots, t_{n+1})^d$. The surjective morphism $A[t_1, \ldots, t_{n+1}][s_1, \ldots, s_{n+1}] \to S$ of such algebras sending s_i to t_i in degree 1 induces a closed immersion $Y \hookrightarrow \mathbf{P}^n_{\mathbf{A}^{n+1}} = \mathbf{P}^n \times \mathbf{A}^{n+1}$ over $\mathbf{A}^{n+1}$. We shall use q to denote the morphism $Y \to \mathbf{P}^n$, and we claim that Y is isomorphic over $\mathbf{P}^n$ to $\mathbf{V}(\mathcal{O}_{\mathbf{P}^n}(1))$. First we note that q is affine since the preimage of the standard affine open $D_+(s_i)$ of $\mathbf{P}^{n-1}$ is the affine open $D_+(t_i)$ of Y, where t_i is in degree 1. By the equivalence between affine schemes and quasi-coherent algebras, we need to show that there is an isomorphism of $\mathbf{P}^n$-algebras

$$\psi \colon S(\mathcal{O}_{\mathbf{P}^n}(1)) \cong q_*(\mathcal{O}_Y).$$

Thus we construct a morphism of $\mathbf{P}^n$-modules $\phi \colon \mathcal{O}_{\mathbf{P}^n}(1) \to q_*(\mathcal{O}_Y)$. On a standard open set $D_+(s_i)$, ϕ is given by the map of $A[s_1, \ldots, s_{n+1}]_{(s_i)}$-modules

$$\phi_i \colon A[s_1, \ldots, s_{n+1}](1)_{(s_i)} \to S_{(t_i)}, \quad s_j s_k / s_i \mapsto t_j t_k / t_i.$$

These maps manifestly coincide over the intersections $D_+(s_i) \cap D_+(s_j)$ and therefore induce our map ϕ. Then over the affine patch $D_+(s_i)$, ψ is the morphism of rings

$$S(A[s_1, \dots, s_{n+1}](1)_{(s_i)}) \to S_{(t_i)}$$

adjoint to ϕ_i, which is easily seen to be an isomorphism. Moreover, the ideal of elements of positive degrees on the left corresponds to the ideal $I_{(t_i)}$ of $S_{(t_i)}$, where I is the graded ideal $\bigoplus_{d \geq 0} (t_1, \dots, t_{n+1})^{d+1}$ of S. This is precisely the kernel of the map from S to its associated graded which corresponds to the closed immersion $Z_{\mathcal{J}} \hookrightarrow Y$. This gives the commutative square (8). The second claim is seen to hold by direct inspection: if we identify $\mathbf{A}^n$ with $B(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}) = \operatorname{Proj}(A[t_1, \dots, t_n][s])$ as in the construction of g_i, each map in (9) is the Proj of a map of graded A-algebras, and their composition is the map $A[s_1, \dots, s_{n+1}] \to A[t_1, \dots, t_n][s]$ sending s_i to $t_i s$ for $i \neq n+1$ and s_{n+1} to s. This clearly corresponds to the $(n+1)$th affine patch $\mathbf{A}^n$ of $\mathbf{P}^n$. [proof of (10)]

Now, for general Z, we apply the previous construction over each affine open set $U \subset Z$, and the obvious fact that the morphism ϕ_i is natural in A shows that the isomorphisms constructed over U restricts to the ones over V if $V \subset U$ are affine, so that they can be glued to give isomorphisms $B(\mathcal{J}) \to \mathbf{V}(\mathcal{O}_{\mathbf{P}^n_Z}(1))$ and $Z_{\mathcal{J}} \to \mathbf{P}^n_Z$. The commutativity of (8) and the other claims automatically hold because they are local conditions on Z. □

Proposition 2.13. *If Z is smooth over S and i is the zero section $Z \hookrightarrow \mathbf{A}^n_Z$ for some $n \geq 0$, then γ_i and δ_i are $\mathbf{A}^1$-equivalences.*

Proof. Since $\mathcal{O}_{\mathbf{P}^n_Z}(1)$ is an invertible $\mathbf{P}^n_Z$-module, the projection $\mathbf{V}(\mathcal{O}_{\mathbf{P}^n_Z}(1)) \to \mathbf{P}^n_Z$ is an $\mathbf{A}^1$-equivalence by Proposition 2.1. By Lemma 2.12, the map $q\colon B(\mathcal{J}) \to \mathbf{P}^n_Z$ is an $\mathbf{A}^1$-equivalence and $q^{-1}(\mathbf{P}^n_Z - Z) = B(\mathcal{J}) - f(Z \times \mathbf{A}^1)$, so that $B(\mathcal{J}) - f(Z \times \mathbf{A}^1) \to \mathbf{P}^n_Z - Z$ is again a vector bundle and hence an $\mathbf{A}^1$-equivalence. By left properness, the induced morphism

$$\epsilon\colon B(\mathcal{J})/(B(\mathcal{J}) - f(Z \times \mathbf{A}^1)) \to \mathbf{P}^n_Z/(\mathbf{P}^n_Z - Z)$$

is an $\mathbf{A}^1$-equivalence of Nisnevich sheaves. Since the composition $\mathbf{P}^n_Z \cong Z' \hookrightarrow B(\mathcal{J}) \to \mathbf{P}^n_Z$ is the identity, $\epsilon \circ \gamma_i$ is the map $\mathbf{P}^n_Z/\mathbf{P}^{n-1}_Z \to \mathbf{P}^n_Z/(\mathbf{P}^n_Z - Z)$ considered in Lemma 2.10 and was shown there to be an $\mathbf{A}^1$-equivalence. This shows that γ_i is an $\mathbf{A}^1$-equivalence.

Also by Lemma 2.12, the composition of $g_i\colon X \to B(\mathcal{J})$ and q is the inclusion $\mathbf{A}^n_Z \hookrightarrow \mathbf{P}^n_Z$ considered in Proposition 2.6, and this proposition shows that $\epsilon \circ \delta_i$ is an isomorphism. Thus, δ_i is also an $\mathbf{A}^1$-equivalence. □

Lemma 2.14. *Let*

$$\begin{tikzcd} Z' \arrow[r, hook, "i'"] \arrow[d, "u"'] & X' \arrow[d, "v"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

be a Nisnevich morphism $i' \to i$ in $\mathcal{I}\mathrm{mm}_{\mathrm{lfp}}$. Then γ_i (resp. δ_i) is an $\mathbf{A}^1$-equivalence if and only if $\gamma_{i'}$ (resp. $\delta_{i'}$) is.

Proof. By naturality of γ and δ, we have a commutative diagram

$$\begin{tikzcd} Z_{\mathcal{J}'}/Z_{\mathcal{I}'} \arrow[r, "\gamma_{i'}"] \arrow[d] & B(\mathcal{J}')/(B(\mathcal{J}') - f_{i'}(Z' \times \mathbf{A}^1)) \arrow[d] & X'/(X' - i'(Z')) \arrow[l, "\delta_{i'}"'] \arrow[d] \\ Z_{\mathcal{J}}/Z_{\mathcal{I}} \arrow[r, "\gamma_i"'] & B(\mathcal{J})/(B(\mathcal{J}) - f_i(Z \times \mathbf{A}^1)) & X/(X - i(Z)) \arrow[l, "\delta_i"] \end{tikzcd} \tag{11}$$

and we claim that the three vertical maps are isomorphism of sheaves, which will prove the lemma. Since v is flat, $v^*(\mathcal{S}_{\mathcal{I}}) = \mathcal{S}_{\mathcal{I}'}$, so that

$$(i')^*(\mathcal{S}_{\mathcal{I}'}) = (i')^* v^*(\mathcal{S}_{\mathcal{I}}) = u^* i^*(\mathcal{S}_{\mathcal{I}}),$$

and hence (since u is an isomorphism) the morphism $Z_{\mathcal{J}'} = \mathrm{Proj}((i')^*(\mathcal{S}_{\mathcal{J}'})) \to Z_{\mathcal{J}} = \mathrm{Proj}(i^*(\mathcal{S}_{\mathcal{J}}))$ is an isomorphism. Similarly, using that $v \times \mathbf{A}^1$ is flat, we have that $Z_{\mathcal{J}'} \to Z_{\mathcal{J}}$ is an isomorphism, and so the first vertical map in (11) is an isomorphism. The equality $(v \times \mathbf{A}^1)^*(\mathcal{S}_{\mathcal{J}}) = \mathcal{S}_{\mathcal{J}'}$ also implies that the square

$$\begin{tikzcd} B(\mathcal{J}') \arrow[r] \arrow[d] & B(\mathcal{J}) \arrow[d] \\ X' \times \mathbf{A}^1 \arrow[r] & X \times \mathbf{A}^1 \end{tikzcd}$$

is actually cartesian, and in particular $B(\mathcal{J}') \to B(\mathcal{J})$ is étale. Using that u is an isomorphism, we see at once that

$$\begin{tikzcd} B(\mathcal{J}') - f_{i'}(Z' \times \mathbf{A}^1) \arrow[r] \arrow[d] & B(\mathcal{J}') \arrow[d] \\ B(\mathcal{J}) - f_i(Z \times \mathbf{A}^1) \arrow[r] & B(\mathcal{J}) \end{tikzcd}$$

is an elementary distinguished square. By Proposition 1.3, the second vertical map in (11) is an isomorphism. The last vertical map in (11) is taken care of similarly: the condition of the theorem implies that the obvious square is elementary distinguished and we use Proposition 1.3. □

Lemma 2.15. *Let $\{U_1, \ldots, U_n\}$ be an open covering of X and assume that $i \in \mathcal{I}\mathrm{mm}_{\mathrm{lfp}}$. If $\gamma_{i \times_X U_\nu}$ (resp. $\delta_{i \times_X U_\nu}$) is an $\mathbf{A}^1$-equivalence for $1 \leq \nu \leq n$, then γ_i (resp. δ_i) is an $\mathbf{A}^1$-equivalence.*

Proof. Let $X_0 = \coprod_i U_i$ and $Z_0 = \coprod_i U_i \times_X Z$, and note that we have a cartesian square

$$\begin{tikzcd} Z_0 \arrow[r, "i_0"] \arrow[d, "u"'] & X_0 \arrow[d, "v"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

where v is flat. Let $X_\bullet$ (resp. $Z_\bullet$) denote the Čech resolution of v (resp. of u). As a simplicial scheme over X, $X_\bullet$ is simply the 0-coskeleton of $X_0 \in \mathcal{S}\mathrm{ch}/X$. By Proposition 2.3 the comma category $\mathcal{I}\mathrm{mm}_{\mathrm{fl}}/i$ has finite limits and the forgetful functor $\mathcal{I}\mathrm{mm}_{\mathrm{fl}}/i \to \mathcal{S}\mathrm{ch}/Z \times \mathcal{S}\mathrm{ch}/X$ creates them. It follows that the degreewise closed immersion $i_\bullet\colon Z_\bullet \hookrightarrow X_\bullet$ induced by $i_0\colon Z_0 \hookrightarrow X_0$ is a simplicial object in $\mathcal{I}\mathrm{mm}_{\mathrm{fl}}/i$ (the 0-coskeleton of i_0). We can therefore apply to it degreewise the functors in (6), and we obtain a commutative diagram of pointed simplicial sheaves

$$\begin{tikzcd} P_2(i_\bullet) \arrow[r, "\gamma_{i_\bullet}"] \arrow[d] & \tilde{Q}(i_\bullet) \arrow[d] & Q(i_\bullet) \arrow[l, "\delta_{i_\bullet}"'] \arrow[d] \\ P_2(i) \arrow[r, "\gamma_i"'] & \tilde{Q}(i) & Q(i). \arrow[l, "\delta_i"] \end{tikzcd}$$

Invoking Proposition 2.9 and right properness, $\gamma_{i_\bullet}$ and $\delta_{i_\bullet}$ are degreewise $\mathbf{A}^1$-equivalences, whence $\mathbf{A}^1$-equivalences of simplicial sheaves. Proposition 2.9 also implies that these functors commute with the formation of coskeletons, so that the three vertical maps are Čech resolutions and in particular equivalences of simplicial sheaves. Thus, γ_i and δ_i are $\mathbf{A}^1$-equivalences. □

The following proposition concludes the proof of Theorem 2.8.

Proposition 2.16. *If S is Noetherian and X and Z are smooth and of finite type over S, then γ_i and δ_i are $\mathbf{A}^1$-equivalences.*

Proof. By [**SGAI**, II.4.9] (implicit in the hypotheses of this result is that all schemes are locally Noetherian), there exists an open covering $\{U_1, \ldots, U_n\}$ of X such that each closed immersion

$i_\nu\colon U_\nu \times_X Z \to U_\nu$ fits into a cartesian square

$$\begin{tikzcd} U_\nu \times_X Z \arrow[r, "i_\nu"] \arrow[d, "u"'] & U_\nu \arrow[d, "v"] \\ \mathbf{A}^r_S \arrow[r] & \mathbf{A}^r_S \times \mathbf{A}^s \end{tikzcd}$$

for some $r, s \geq 0$, where v is étale and the lower map is the zero section. On the other hand, the zero sections also form a cartesian square

$$\begin{tikzcd} U_\nu \times_X Z \arrow[r, "i'_\nu"] \arrow[d, "u"'] & (U_\nu \times_X Z) \times \mathbf{A}^s \arrow[d, "u\times\mathbf{A}^s"] \\ \mathbf{A}^r_S \arrow[r] & \mathbf{A}^r_S \times \mathbf{A}^s \end{tikzcd}$$

by Proposition 2.5, and $\gamma_{i'_\nu}$ and $\delta_{i'_\nu}$ are $\mathbf{A}^1$-equivalences by Proposition 2.13. By Lemma 2.4, i_ν and i'_ν are related by a zigzag of Nisnevich morphisms, so by Lemma 2.14 γ_{i_ν} and δ_{i_ν} are also $\mathbf{A}^1$-equivalences. Now by Lemma 2.15, γ_i and δ_i are $\mathbf{A}^1$-equivalences. $\square$

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