K-THEORY OF DUALIZABLE CATEGORIES
(AFTER A. EFIMOV)

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We explain the definition of the K-theory of "large" stable ∞-categories, due to Alexander Efimov. These notes are based on a talk given by Efimov at the CATS5 conference in Lisbon in October 2018.

We start by fixing some terminology. We shall refer to ∞-categories as categories. We write $\mathcal{P}\mathrm{r}$ for the category of presentable categories and colimit-preserving functors, $\mathcal{P}\mathrm{r}^{\mathrm{dual}} \subset \mathcal{P}\mathrm{r}$ for the subcategory of dualizable objects and right-adjointable morphisms (with respect to the symmetric monoidal and 2-categorical stuctures of $\mathcal{P}\mathrm{r}$), and $\mathcal{P}\mathrm{r}^{\mathrm{cg}} \subset \mathcal{P}\mathrm{r}$ for the subcategory of compactly generated categories and compact functors (i.e., functors whose right adjoints preserve filtered colimits). The subscript "St" will denote the corresponding full subcategories consisting of stable categories. A *localization sequence* in any such category will mean a cofiber sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ where $\mathcal{A} \to \mathcal{B}$ is fully faithful.

It is well known that $\mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$ is a full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$. In fact:

Theorem 1 (Lurie)**.** *For $\mathcal{C}$ a stable presentable category, the following are equivalent:*

(1) *$\mathcal{C}$ is dualizable in $\mathcal{P}\mathrm{r}$.*
(2) *$\mathcal{C}$ is a retract in $\mathcal{P}\mathrm{r}$ of a compactly generated category.*
(3) *The colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{y}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.*

Note that the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is left adjoint to the Yoneda embedding $y\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$, which is fully faithful. Hence, the functor $\hat{y}$ is also fully faithful. When $\mathcal{C}$ is compactly generated, $\hat{y}$ is the Ind-extension of the inclusion $\mathcal{C}^\omega \subset \mathcal{C}$.

Definition 2. Let $\mathcal{C}$ be a presentable stable category. The *Calkin category* $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ is the kernel of the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

Proposition 3. *Suppose $\mathcal{C} \in \mathcal{P}\mathrm{r}$ is stable and dualizable. Then:*

(1) $\mathrm{Calk}(\mathcal{C}) = \mathrm{Ind}(\mathrm{Calk}(\mathcal{C})^\omega)$.
(2) *The inclusion $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ admits a left adjoint*
$$\Phi\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Calk}(\mathcal{C})$$
that preserves compact objects.
(3) *There is a localization sequence of cocomplete stable categories*
$$\mathcal{C} \underset{\mathrm{colim}}{\overset{\hat{y}}{\leftrightarrows}} \mathrm{Ind}(\mathcal{C}) \overset{\Phi}{\leftrightarrows} \mathrm{Calk}(\mathcal{C}).$$

Proof. The left adjoint Φ is the cofiber of the counit transformation $\hat{y} \circ \mathrm{colim} \to \mathrm{id}$. It preserves compact objects because its right adjoint preserves colimits. Since Φ is essentially surjective, $\mathrm{Calk}(\mathcal{C})$ is generated under colimits by $\Phi(\mathcal{C}) \subset \mathrm{Calk}(\mathcal{C})^\omega$, and the first assertion follows. □

Note that $\mathrm{Calk}(\mathcal{C})^\omega$ is a locally small but *large* category, so that $\mathrm{Calk}(\mathcal{C})$ is not presentable (unless $\mathcal{C} = 0$).

Definition 4. Let $\mathcal{C} \in \mathcal{P}\mathrm{r}$ be stable and dualizable. The *continuous K-theory* of $\mathcal{C}$ is the space
$$\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) = \Omega\mathrm{K}(\mathrm{Calk}(\mathcal{C})^\omega).$$

Lemma 5. *If $\mathcal{C}$ is compactly generated, then $\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) = \mathrm{K}(\mathcal{C}^\omega)$.*

Proof. In this case, the localization sequence of Proposition 3 is Ind of the sequence
$$\mathcal{C}^\omega \hookrightarrow \mathcal{C} \to \mathrm{Calk}(\mathcal{C})^\omega.$$
Since $\mathrm{K}(\mathcal{C}) = 0$, the result follows from the localization theorem in K-theory. □

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More generally, we can define a continuous version of any localizing invariant. In the case of K-theory, we made use of the fact that K-theory is defined on the large category $\mathrm{Calk}(\mathcal{C})^\omega$, but this is not the case for an abstract localizing invariant. Thus, we will need to use a presentable version of the Calkin category, which depends on the choice of a large enough regular cardinal.

Definition 6. Let $\mathcal{C}$ be a presentable stable category and κ a regular cardinal. The *κ-Calkin category* $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$ is the kernel of the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$.

If $\mathcal{C} \in \mathcal{P}\mathrm{r}$ is dualizable, there exists a regular cardinal κ such that the fully faithful functor $\hat{y}\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ lands in $\mathrm{Ind}(\mathcal{C}^\kappa)$. We shall then say that $\mathcal{C}$ is *κ-dualizable*, and we let $\mathcal{P}\mathrm{r}^{\kappa\text{-dual}} \subset \mathcal{P}\mathrm{r}^{\mathrm{dual}}$ be the full subcategory spanned by the κ-dualizable categories. We have

$$\mathcal{P}\mathrm{r}^{\mathrm{dual}} = \bigcup_\kappa \mathcal{P}\mathrm{r}^{\kappa\text{-dual}} \quad \text{and} \quad \mathcal{P}\mathrm{r}^{\kappa\text{-dual}} \subset \mathcal{P}\mathrm{r}^{\kappa\text{-cg}}.$$

For example, a stable presentable category is ω-dualizable if and only if it is compactly generated. Given $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\kappa\text{-dual}}_{\mathrm{St}}$, we have a localization sequence

$$\mathcal{C} \underset{\mathrm{colim}}{\overset{\hat{y}}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}^\kappa) \overset{\Phi_\kappa}{\rightleftarrows} \mathrm{Calk}_\kappa(\mathcal{C}) \tag{7}$$

in $\mathcal{P}\mathrm{r}$, where Φ_κ is the restriction of Φ, and this diagram is functorial in $\mathcal{C}$. Moreover, it is easy to check that $\mathrm{Calk}_\kappa\colon \mathcal{P}\mathrm{r}^{\kappa\text{-dual}}_{\mathrm{St}} \to \mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$ preserves localization sequences.

Let $\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}}$ be the category of small stable idempotent complete categories and exact functors. Recall that Ind-completion induces an isomorphism $\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$. If $\mathcal{T}$ is a category, we shall say that a functor $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ is a *localizing invariant* if it preserves final objects and sends localization sequences to fiber sequences.

Lemma 8. *Let $\mathcal{T}$ be a category and $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ a localizing invariant. Let $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}$ be dualizable and let $\kappa \leq \lambda$ be uncountable regular cardinals such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$. Then the inclusion $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Calk}_\lambda(\mathcal{C})$ induces an isomorphism*

$$F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega) \simeq F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega).$$

Proof. Applying Calk_κ to the localization sequence (7) for λ gives a localization sequence

$$\mathrm{Calk}_\kappa(\mathcal{C}) \to \mathrm{Calk}_\kappa(\mathrm{Ind}(\mathcal{C}^\lambda)) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C})).$$

On the other hand, applying (7) to $\mathrm{Calk}_\lambda(\mathcal{C})$ gives a localization sequence

$$\mathrm{Calk}_\lambda(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Calk}_\lambda(\mathcal{C})^\kappa) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C})).$$

Hence, we obtain isomorphisms

$$F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega) \simeq \Omega F(\mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C}))^\omega) \simeq F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega),$$

natural in λ. This proves the claim. ☐

Given Lemma 8, the following definition makes sense:

Definition 9. Let $\mathcal{T}$ be a category and $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ a localizing invariant. The *continuous extension* of F is the functor $F_{\mathrm{cont}}\colon \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathcal{T}$ defined by

$$F_{\mathrm{cont}}(\mathcal{C}) = \Omega F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega),$$

where κ is any uncountable regular cardinal such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$.

More formally, we can consider $(\mathrm{Calk}_\kappa)_\kappa$ as a functor

$$\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \widehat{\mathrm{Ind}}(\mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}) \simeq \widehat{\mathrm{Ind}}(\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}})$$

and compose it with the $\widehat{\mathrm{Ind}}$-extension of ΩF, the result landing in the subcategory $\mathcal{T} \subset \widehat{\mathrm{Ind}}(\mathcal{T})$ by Lemma 8. In the case of K-theory, Definition 9 agrees with Definition 4 since $\mathrm{Calk}(\mathcal{C})^\omega = \mathrm{colim}_\kappa \mathrm{Calk}_\kappa(\mathcal{C})^\omega$ and K-theory commutes with filtered colimits.

We shall say that a functor $F\colon \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathcal{T}$ is a *localizing invariant* if it preserves final objects and sends localization sequences to fiber sequences.

Theorem 10 (Efimov)**.** *Let $\mathcal{T}$ be a category. The functor*

$$\mathrm{Fun}(\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}, \mathcal{T}) \to \mathrm{Fun}(\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}}, \mathcal{T}), \quad F \mapsto F \circ \mathrm{Ind},$$

restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{\mathrm{cont}}$. In particular, if $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$, then $F_{\mathrm{cont}}(\mathcal{C}) = F(\mathcal{C}^\omega)$.

Proof. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a localization sequence in $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$. Then for large enough κ we have an induced localization sequence

$$\mathrm{Calk}_\kappa(\mathcal{A}) \to \mathrm{Calk}_\kappa(\mathcal{B}) \to \mathrm{Calk}_\kappa(\mathcal{C}).$$

It follows that F_{cont} is a localizing invariant. The proof of Lemma 5 shows that $F_{\mathrm{cont}} \circ \mathrm{Ind} \simeq F$, functorially in F. To $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$ we can associate the filtered diagram of localization sequences

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathrm{Calk}_\kappa(\mathcal{C}), \quad \kappa \gg 0,$$

which immediately gives a functorial isomorphism $F \simeq (F \circ \mathrm{Ind})_{\mathrm{cont}}$. □

Example 11. An example of a localizing invariant $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathrm{Sp}$ is the functor sending a dualizable category to its Euler characteristic. In fact, this functor is the continuous extension of $\mathrm{THH}\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathrm{Sp}$, since $\mathrm{THH}(\mathcal{C})$ is the Euler characteristic of $\mathrm{Ind}(\mathcal{C})$ in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.

Example 12. Theorem 10 implies the following:

- $\mathrm{THH}_{\mathrm{cont}}\colon \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathrm{Sp}$ factors through the stable category CycSp of cyclotomic spectra;
- $\mathrm{TP}_{\mathrm{cont}}(\mathcal{C}) \simeq \mathrm{THH}_{\mathrm{cont}}(\mathcal{C})^{t\mathbb{T}}$ and $\mathrm{TC}_{\mathrm{cont}}(\mathcal{C})$ is the mapping spectrum $\mathrm{Map}_{\mathrm{CycSp}}(\mathbf{1}, \mathrm{THH}_{\mathrm{cont}}(\mathcal{C}))$;
- $\Omega^\infty(\mathbb{K}_{\mathrm{cont}}) \simeq \mathrm{K}_{\mathrm{cont}}$, where $\mathbb{K}$ is nonconnective K-theory;
- the cyclotomic trace $\mathbb{K} \to \mathrm{TC}$ extends uniquely to a transformation $\mathbb{K}_{\mathrm{cont}} \to \mathrm{TC}_{\mathrm{cont}}$.

One immediately recovers the following excision theorem of Tamme:

Corollary 13 (Tamme)**.** *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r, "g"] & \mathcal{D} \end{tikzcd}$$

be a commutative square in $\mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$, cartesian in $\mathcal{C}\mathrm{at}$, such that the right adjoint of g is fully faithful. For any stable category $\mathcal{T}$ and any localizing invariant $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$, the induced square

$$\begin{tikzcd} F(\mathcal{A}^\omega) \arrow[r] \arrow[d] & F(\mathcal{B}^\omega) \arrow[d] \\ F(\mathcal{C}^\omega) \arrow[r] & F(\mathcal{D}^\omega) \end{tikzcd}$$

is cartesian.

Proof. Note that the right adjoint of f is also fully faithful. The kernels of f and g are dualizable and isomorphic. Since $\mathcal{T}$ is stable, F_{cont} takes the given square to a cartesian square. □

For the next lemma, note that each subcategory $\mathcal{P}\mathrm{r}^{\kappa\text{-dual}}_{\mathrm{St}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is closed under small colimits, hence $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is closed under small colimits.

Lemma 14. *Let $\mathcal{T}$ be a category, $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ a localizing invariant, and $\mathcal{K}$ a small category. Then F preserves $\mathcal{K}$-indexed colimits if and only if F_{cont} preserves $\mathcal{K}$-indexed colimits.*

Proof. This follows easily from (7) and the fact that $\mathrm{Ind}\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{P}\mathrm{r}_{\mathrm{St}}$ preserves colimits. □

Theorem 15. *Let X be a locally compact Hausdorff topological space, $\mathcal{C}$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on X.*

(1) (Lurie) $\mathrm{Mod}_R(\mathrm{Shv}(X, \mathcal{C}))$ *is dualizable in* $\mathcal{P}\mathrm{r}$.

(2) (Efimov) *Suppose that* $\mathrm{Shv}(X)$ *is hypercomplete (for example,* X *is a topological manifold). Let* $\mathcal{T}$ *be a stable compactly generated category and* $F\colon \mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}} \to \mathcal{T}$ *a localizing invariant that preserves filtered colimits. Then*

$$F_{\mathrm{cont}}(\mathrm{Mod}_R(\mathrm{Shv}(X,\mathcal{C}))) \simeq \Gamma_{\mathrm{c}}(X, F_{\mathrm{cont}}(\mathrm{Mod}_R(\mathcal{C}))).$$

In particular,

$$F_{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n,\mathcal{C})) \simeq \Omega^n F_{\mathrm{cont}}(\mathcal{C}).$$

Proof. Let $\mathcal{K}(X)$ denote the poset of compact subsets of X. We write $K \Subset L$ if $K \subset U \subset L$ for some open subset $U \subset X$. There is a fully faithful embedding

$$e\colon \mathrm{Shv}(X,\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}), \quad e(\mathcal{F})(K) = \operatorname*{colim}_{K\subset U} \mathcal{F}(U),$$

identifying $\mathrm{Shv}(X,\mathcal{C})$ with the subcategory of functors $\mathcal{G}\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{C}$ such that

(a) $\mathcal{G}(\emptyset) = *$;

(b) for every $K, L \in \mathcal{K}(X)$, the following square is cartesian:

$$\begin{tikzcd}
\mathcal{G}(K\cup L) \arrow[r] \arrow[d] & \mathcal{G}(K) \arrow[d] \\
\mathcal{G}(L) \arrow[r] & \mathcal{G}(K\cap L);
\end{tikzcd}$$

(c) for every $K \in \mathcal{K}(X)$, the canonical map $\mathcal{G}(K) \to \operatorname{colim}_{K\Subset L} \mathcal{G}(L)$ is an isomorphism.

Since $\mathcal{C}$ is stable, e preserves colimits. Let $\mathcal{E} \subset \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C})$ be the full subcategory of functors satisfying conditions (a) and (b). Define the endofunctor $\mathcal{G} \mapsto \mathcal{G}^\dagger$ of $\mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C})$ by the formula

$$\mathcal{G}^\dagger(K) = \operatorname*{colim}_{K\Subset L} \mathcal{G}(L).$$

One can easily show that the functor $\mathcal{G} \mapsto \mathcal{G}^\dagger$ sends $\mathcal{E}$ to itself. It follows that the inclusion $e(\mathrm{Shv}(X,\mathcal{C})) \subset \mathcal{E}$ has a right adjoint given by $\mathcal{G} \mapsto \mathcal{G}^\dagger$, which exhibits $\mathrm{Shv}(X,\mathcal{C})$ as a retract of $\mathcal{E}$ in $\mathcal{P}\mathrm{r}$. Since $\mathcal{E}$ is clearly dualizable, this shows that $\mathrm{Mod}_R(\mathrm{Shv}(X,\mathcal{C}))$ is dualizable.

Consider the functor $\tilde{\mathcal{C}}\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}}$ defined by

$$\tilde{\mathcal{C}}(K) = \mathrm{Shv}(K,\mathcal{C}).$$

Since F_{cont} is a stable localizing invariant that preserves filtered colimits (by Lemma 14), $F_{\mathrm{cont}} \circ \mathrm{Mod}_{e(R)}(\tilde{\mathcal{C}})$ satisfies conditions (a)–(c). Hence, $F_{\mathrm{cont}} \circ \mathrm{Mod}_{e(R)}(\tilde{\mathcal{C}}) = e(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{Shv}(X,\mathcal{T})$. The obvious map

$$F_{\mathrm{cont}}(\mathrm{Mod}_R(\mathcal{C})) \to \mathcal{F}$$

induces isomorphisms on stalks. Since $\mathrm{Shv}(X)$ is hypercomplete and $\mathcal{T}$ is compactly generated, it exhibits $\mathcal{F}$ as the sheafification of $F_{\mathrm{cont}}(\mathrm{Mod}_R(\mathcal{C}))$. It remains to show that

$$\Gamma_{\mathrm{c}}(X,\mathcal{F}) \simeq F_{\mathrm{cont}}(\mathrm{Mod}_R(\mathrm{Shv}(X,\mathcal{C}))). \tag{16}$$

This is obvious if X is compact. In general, choose a compactification $j\colon X \hookrightarrow \bar{X}$ with closed complement $i\colon X_\infty \hookrightarrow \bar{X}$. Let $\bar{\mathcal{F}}$ be the sheaf on $\bar{X}$ such that $e(\bar{\mathcal{F}}) = F_{\mathrm{cont}} \circ \mathrm{Mod}_{e(j_*(R))}(\tilde{\mathcal{C}})$. Then $j^*(\bar{\mathcal{F}}) \simeq \mathcal{F}$ and hence there is a fiber sequence

$$\Gamma_{\mathrm{c}}(X,\mathcal{F}) \to \Gamma(\bar{X},\bar{\mathcal{F}}) \to \Gamma(X_\infty, i^*\bar{\mathcal{F}}).$$

On the other hand, the sequence

$$\mathrm{Mod}_R(\mathrm{Shv}(X,\mathcal{C})) \xrightarrow{j_!} \mathrm{Mod}_{j_*(R)}(\mathrm{Shv}(\bar{X},\mathcal{C})) \xrightarrow{i^*} \mathrm{Mod}_{i^*j_*(R)}(\mathrm{Shv}(X_\infty,\mathcal{C}))$$

is a localization sequence in $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}}$. Applying F_{cont} proves (16). □

Remark 17. Let $\mathcal{E}$ be a small stable rigid symmetric monoidal category, for example Perf_k for some E_∞-ring spectrum k. Then Theorem 10 remains true if we replace $\mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}}$ by $\mathrm{Mod}_{\mathcal{E}}(\mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}})$ and $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}}$ by $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{E})}(\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}})$. The rigidity of $\mathcal{E}$ ensures that $\hat{y}$ and Φ are $\mathcal{E}$-linear functors. A similar generalization of Theorem 15 holds, replacing R with an $\mathrm{Alg}(\mathrm{Ind}(\mathcal{E}))$-valued sheaf on X.