Chern character and derived algebraic geometry

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Contents

Introduction

The subject of this Master thesis was motivated by the article [**TV08**] from B. Toën and G. Vezzosi. In this article they define categorical sheaves on schemes and use a construction from derived algebraic geometry to define the Chern character of such a sheaf. The original goal of this thesis was to go as far as possible towards a precise understanding of these constructions. In the end, the focus has shifted to the related construction of the Chern character of an "ordinary" sheaf, culminating in Chapter 4 with a computation of this character for vector bundles on derived affine schemes, and only a very small part of this text is dedicated to the study of categorical sheaves themselves.

The first chapter discusses Hochschild homology, cyclic homology, and the Chern character in the classical context of algebras. It was written first and in certain places is only remotely related to the remaining chapters. The material here is also much older. I believe however that some parts of the presentation are new, namely, the explicit formula for the map on Hochschild complexes induced by a bimodule and the emphasis on Morita invariance which is used to lift the Chern character to negative cyclic homology (and accessorily to give a simple proof that Hochschild and cyclic homology preserve finite products).

The second chapter reviews the theory of stacks on model categories as presented in [**HAGI**, §4]. The third chapter develops the basic setup for derived algebraic geometry in the language of stacks on model categories. We give detailed proofs of two results from [**HAGII**]: the flat descent theorem (Theorem 24) and a characterization of dualizable objects in the homotopy category of simplicial modules (Theorem 37).

Chapter 4 comes back to Hochschild and cyclic homology and gives geometric interpretations of these constructions in term of the loop space of a derived stack. In §4.3, we give a detailed construction of the Chern character of a vector bundle on a derived stack, and we successfully prove the claim made in [**TV08**] that it is compatible with the classical Chern character of Chapter 1.

Chapter 5 was originally meant to be an exposition of the construction of the Chern character for categorical sheaves outlined in [**TV08**]. Categorical sheaves are defined there as sheaves of differential graded categories on derived stacks. As it appeared that most of the results beyond the basic homotopy theory of differential graded categories would be conjectural and that there was not enough time to tackle their proofs, I decided to modify slightly the construction by replacing dg categories with simplicial categories. The two approaches are not unrelated since it is proved in [**Tab07a**] that the homotopy theories of nonnegatively differential graded categories and of linear simplicial categories are equivalent through a generalized normalization functor; so the only added value of dg categories is the possibility of having unbounded complexes of morphisms. There are similarly two approaches to derived stacks, one that uses dg rings and another that uses simplicial rings, and since this text presents the simplicial approach to derived stacks it is only natural to use the simplicial approach to categorical sheaves as well. Indeed, in the "mixed" setting of [**TV08**], one has to use the normalization functor to be able to speak of a dg category over a simplicial ring. All the propositions of Chapter 5 stated as "conjectures" are known to be true if one replaces simplicial categories by dg categories (see [**Toë06b**, **TV07**, **Tab07b**]), and I believe that many of them can in fact be derived as consequences of these known results using the above equivalence. The construction of the Chern character itself is almost word for word the same as that presented in Chaper 4.

Unfortunately, there remain a few unproved results in the text, the most important one being Theorem 48. The other ones are three small technical results, namely

- Proposition 16,
- an argument in the proof of Proposition 22, and
- Lemma 27.

all of which are claimed to be true in [**HAGII**].

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Notations, terminology, etc.

Our conventions regarding categories are mostly self-explanatory. We use the definition of model category of [**Hir03**]: it has functorial factorizations but they need not be part of the model structure. Our notions of monoidal model category, (closed) C-module (C a symmetric monoidal category), and C-model category (C a symmetric monoidal model category) are those defined in [**Hov99**, §4]. The following rule is applied to distinguish between the multitude of hom objects: the hom sets of a category C are always denoted by $\mathsf{C}(x, y)$, and the notations $\mathrm{Map}(x, y)$ and $\mathrm{Hom}(x, y)$ are used respectively for the simplicial mapping spaces of a simplicial category (and in a few instances for enriched hom sets of an enriched category) and the internal hom objects of a closed monoidal category. The category of simplicial objects (resp. cosimplicial objects) in C is denoted by sC (resp. by cC). The objects of the simplicial index category Δ will simply be denoted by 0, 1, 2, etc. If $n \geq 0$, $\mathsf{s}_n\mathsf{C}$ denotes the category of functors from the full subcategory of Δ^{op} spanned by 0, ..., n to C. Decorated arrows follow no strict convention. For instance, an arrow $\twoheadrightarrow$ may denote either an epimorphism or a fibration depending on the context.

Although universes are never mentioned in the text, they are implicitely used starting from Chapter 2 in the following manner. We fix two universes $\mathbb{U} \in \mathbb{V}$ and we assume that the base commutative ring k belongs to $\mathbb{U}$. The base model category C of Chapter 2 is an $\mathbb{U}$-bicomplete category whose set of objects belongs to $\mathbb{V}$ and whose sets of morphisms between any two objects belong to $\mathbb{U}$. In forming the functor category $\mathsf{sSet}^{\mathsf{C}}$ the target category sSet is the category of $\mathbb{V}$-small simplicial sets (i.e. simplicial sets belonging to $\mathbb{V}$). This is extremely important as it allows the values of a functor $\mathsf{C} \to \mathsf{sSet}$ to be as big as the category C itself. In our application to derived algebraic geometry in Chapter 3, sMod_k is the category of $\mathbb{U}$-small simplicial k-modules. The word "small" will always mean $\mathbb{U}$-small (except when it is used informally).

1 The classical Chern character

Let k be a commutative ring and A an associative and unital k-algebra (henceforth simply a k-algebra). Then the *Chern character* of A is a natural map of graded groups

$$\mathrm{ch}_*^-\colon K_*(A) \to HC_*^-(A)$$

where $K_*(A)$ is the K-theory of A and $HC_*^-(A)$ is the negative cyclic homology of A. In this chapter we shall review the definitions and the elementary properties of the objects involved to arrive at the definition of the Chern character in degree 0.

As an example, suppose that k is an algebraically closed field and consider the group algebra $k[G]$ of a finite group G whose order is not a multiple of the characteristic of k. Here, $K_0(k[G])$ is the Grothendieck group of the category of finite-dimensional representations of G (this category being equivalent to that of finitely generated projective $k[G]$-modules), and $HC_0(k[G])$ is the group of complex-valued functions on the conjugacy classes of G. Then if $\rho\colon G \to \mathrm{Aut}(V)$ is a finite-dimensional representation of G over k, $\mathrm{ch}_0(\rho)$ is just the usual character of ρ: for $C \subset G$ a conjugacy class and $g \in C$,

$$\mathrm{ch}_0(\rho)(C) = \mathrm{tr}(\rho(g)).$$

We first recall the definition of the functor K_0. Let A be a k-algebra. We define $K_0(A)$ as the Grothendieck group of the category of finitely generated and projective (f.g.p.) left A-modules. In other words, if $\mu(A)$ is the monoid of isomorphisms classes of f.g.p. left A-modules, with the law of composition induced by the direct sum, then $\mu(A) \to K_0(A)$ is the universal monoid map from $\mu(A)$ to groups. If A is commutative, the tensor product gives $\mu(A)$ the structure of a "semiring" which makes $K_0(A)$ into a commutative ring, and $\mu(A) \to K_0(A)$ is the universal semiring map from $\mu(A)$ to rings. A map of k-algebras $f\colon A \to B$ induces a map of monoids (or semirings) $f_*\colon \mu(A) \to \mu(B)$ by extension of scalars, which lifts to a map of groups (or rings) $f_*\colon K_0(A) \to K_0(B)$ by universality. Thus, K_0 is a covariant functor on Alg_k. The K_0 construction also exhibits a contravariant behaviour if we impose some finiteness condition on maps of algebras. Specifically, if $f\colon A \to B$ is a map of k-algebras that makes B into an f.g.p. left A-module, then restriction of scalars along f induces $f^*\colon \mu(B) \to \mu(A)$, whence $f^*\colon K_0(B) \to K_0(A)$. This makes K_0 into a contravariant functor on a subcategory of Alg_k. In the sequel we shall view K_0 primarily as a covariant functor.

1.1 Hochschild and cyclic homology of algebras

Throughout this section, A denotes an associative and unital algebra over some commutative ring k, and M is an A-bimodule. We first review the definitions of the Hochschild homology and the various cyclic homologies of A.

The *Hochschild complex* $C(A, M)$ of A with values in M is the simplicial k-module with $C_n(A, M) = M \otimes A^{\otimes n}$, with face maps $d_i\colon C_n(A, M) \to C_{n-1}(A, M)$ given by

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & \text{if } i = 0, \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } 1 \le i \le n-1, \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases}$$

and with degeneracy maps $s_i\colon C_n(A, M) \to C_{n+1}(A, M)$ given by

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

The Hochschild homology $H(A, M)$ of A with values in M is then the graded k-module with $H_n(A, M) = \pi_n C(A, M)$.

It is clear that $C(A, M)$ is a functor of M, and it is even a functor from the fibered category of k-algebras and bimodules over them. That is, if N is a B-bimodule, then a pair of maps $f\colon A \to B$, $m\colon M \to f^*N$ induces a simplicial map $C(f, m)\colon C(A, M) \to C(B, N)$, and this construction respects identities and compositions.

When $M = A$, we write $C(A, M) = C(A)$ and $H(A, M) = HH(A)$; $HH(A)$ is the *Hochschild homology of A*. It follows from the functoriality of $C(A, M)$ that $C(A)$ (resp. $HH(A)$) is a functor from k-algebras to simplicial k-modules (resp. to graded k-modules).

Before introducing cyclic homology we recall a categorical construction. Let I be an index category and C a monoidal category with colimits. The tensor product over I of two functors $F\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{C}$ and $G\colon \mathsf{I} \to \mathsf{C}$ is defined as the coend

$$F \otimes_{\mathsf{I}} G = \int^{i \in \mathsf{I}} F(i) \otimes G(i).$$

This is the usual "geometric realization" construction: $F \otimes_{\mathsf{I}} G$ is the coequalizer of the two obvious maps $\coprod_{k \to j} F(j) \otimes G(k) \rightrightarrows \coprod_{i \in \mathsf{I}} F(i) \otimes G(i)$. Suppose moreover that C is abelian and that the tensor product in C is biadditive, e.g. $\mathsf{C} = \mathsf{Mod}_k$. Then $\mathsf{C}^{\mathsf{I}^{\mathrm{op}}}$ and C^{I} are also abelian categories and $\otimes_{\mathsf{I}}$ is a biadditive functor, so we may consider the classical left derived functors of $F \otimes_{\mathsf{I}} ?$ which, when they exist, are denoted $\mathrm{Tor}_n^{\mathsf{I}}(F, ?)$. These are of course bifunctors and reduce to the usual Tor objects when I is a point. For $\mathsf{I} = \mathsf{\Delta}$ and $\mathsf{C} = \mathsf{Mod}_k$ we have the following result (whose proof is detailed from [**Lod92**, §6.2]).

Theorem 1. *The nth homotopy module functor* $\pi_n\colon \mathsf{sMod}_k \to \mathsf{Mod}_k$ *is isomorphic to* $\mathrm{Tor}_n^{\mathsf{\Delta}}(?, k)$*, where k is viewed as a constant cosimplicial k-module.*

Proof. Left derived functors are computed using projective resolutions. Let K_n denote the free cosimplicial k-module generated by the cosimplicial set $\Delta_n = \mathsf{\Delta}(n, ?)$. For $\phi\colon m \to n$, let $K(\phi)\colon K_n \to K_m$ be the morphism of cosimplicial modules induced by precomposition by ϕ. The functor K is then a simplicial cosimplicial k-module with a canonical augmentation $K_0 \to k$ to the constant cosimplicial module k that sends all the generators to 1. Thus, it has an associated augmented chain complex $K_* \to k$ of cosimplicial k-modules. We claim that this is a projective resolution of k.

That each K_n is projective follows from the Mod_k-enriched Yoneda lemma. More precisely, if we apply the free k-module functor to every set of morphisms of $\mathsf{\Delta}$ we obtain a Mod_k-enriched category $k[\mathsf{\Delta}]$ and the cosimplicial k-module K_n is just the functor represented by n on $k[\mathsf{\Delta}]$. The Yoneda lemma then says that the functor $\mathsf{Mod}_k^{\mathsf{\Delta}}(K_n, ?)$ is isomorphic to the "evaluation at n" functor $\mathsf{Mod}_k^{\mathsf{\Delta}} \to \mathsf{Mod}_k$, which is exact. By definition, this means that K_n is projective.

Homology of complexes of functors to abelian categories is computed pointwise, so we must check that each complex

$$\cdots \to K_2(m) \to K_1(m) \to K_0(m) \to k(m) = k$$

is a resolution of k. From the definition of $K(\phi)$ we see that this complex is associated to the augmented simplicial k-module $L \to k$ freely generated by the augmented simplicial set $\Delta^m = \mathsf{\Delta}(?, m) \to *$. This augmented simplicial set admits an "extra degeneracy" $h_n\colon \Delta^m(n) \to \Delta^m(n+1)$ given by $h_{-1}(*)(0) = m$ and

$$h_n(\phi)(j) = \begin{cases} \phi(j) & \text{if } 0 \le j \le n, \\ m & \text{if } j = n+1, \end{cases}$$

for $n \ge 0$. It induces an extra degeneracy on $L \to k$, which is therefore aspherical.

If E is a simplicial k-module, it follows that we can compute $\mathrm{Tor}_*^{\mathsf{\Delta}}(E, k)$ as the homology of the complex $E \otimes_{\mathsf{\Delta}} K_*$. Hence it will suffice to prove that $E \otimes_{\mathsf{\Delta}} K_*$ is naturally isomorphic to the chain complex associated to E (whose homology modules are the homotopy modules of E). We prove more generally that the simplicial k-modules $E \otimes_{\mathsf{\Delta}} K$ and E are naturally isomorphic. Define $\alpha_n\colon E \otimes_{\mathsf{\Delta}} K_n \to E_n$ and $\beta_n\colon E_n \to E \otimes_{\mathsf{\Delta}} K_n$ by $\alpha_n(y \otimes \psi) = E(\psi)(y)$ and $\beta_n(x) = x \otimes \mathrm{id}_n$. It is clear from the coequalizer description of $E \otimes_{\mathsf{\Delta}} K_n$ that these maps are well-defined, natural in E, and inverse to each other. For $\phi\colon m \to n$, we must verify the relations $E(\phi)\alpha_n = \alpha_m(\mathrm{id}_E \otimes_{\mathsf{\Delta}} K(\phi))$ and $(\mathrm{id}_E \otimes_{\mathsf{\Delta}} K(\phi))\beta_n = \beta_m E(\phi)$; the former is trivial and the latter is $x \otimes \phi = E(\phi)(x) \otimes \mathrm{id}_m$ which is true because $\phi = K_m(\phi)(\mathrm{id}_m)$. ∎

This theorem suggests that for index categories I other than Δ, we can define "generalized homotopy groups" of functors $F\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{Mod}_k$ as the k-modules $\mathrm{Tor}_n^{\mathsf{I}}(F,k)$. Cyclic homology is just one example of this idea, when I is the *cyclic category.*

The cyclic category Λ is an extension of Δ with the same objects but having for each n a permutation $n \to n$ of order $n+1$ as an additional morphism, satisfying some relations displayed (in dual form) below. The structure theorem for Λ says that any morphism in Λ factors uniquely as an automorphism followed by a map in Δ. A cyclic (resp. cocylic) object in a category C is a functor $\Lambda^{\mathrm{op}} \to \mathsf{C}$ (resp. $\Lambda \to \mathsf{C}$). It can be shown that defining a cyclic object X amounts to defining objects X_n, $n \geq 0$, and morphisms $d_i\colon X_n \to X_{n-1}$, $s_i\colon X_n \to X_{n+1}$, and $c_n\colon X_n \to X_n$ $(0 \leq i \leq n)$ satisfying the usual simplicial identities as well as

$$\begin{aligned} d_i c_n &= c_{n-1} d_{i-1}, \\ s_i c_n &= c_{n+1} s_{i-1}, \text{ and} \\ c_n^{n+1} &= \mathrm{id}, \end{aligned}$$

where $d_{-1} = c_{n-1}^{-1} d_n$ and $s_{-1} = c_{n+1} s_n$. Here c_n is the image by X of the cyclic permutation $n \to n$ which sends 0 to n, 1 to 0, 2 to 1, etc. (Dually, cocyclic objects are determined by morphisms d^i, s^i, and c_n satisfying the same identities with compositions reversed.)

We endow the Hochschild complex $C(A)$ with a cyclic k-module structure by defining

$$c_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

In this way we obtain a functor from k-algebras with values in cyclic k-modules (in fact, the morphisms $C(f)$ will commute with any permutation of the factors, not only cyclic ones).

We introduce some useful operators on a cyclic object X in a category enriched over abelian groups. We usually write $b_n\colon X_n \to X_{n-1}$ for the differential of the chain complex associated to the underlying simplicial object of X, and we write b'_n for $b_n - (-1)^n d_n = \sum_{i=0}^{n-1} (-1)^i d_i$. The map b' also defines a chain complex since $b'^2 = 0$. There is a signed version of c_n defined by $t_n = (-1)^n c_n$. The averaging operator N_n is $\mathrm{id} + t_n + \cdots + t_n^n$; it satisfies $t_n N_n = N_n t_n = N_n$ and in particular $(\mathrm{id} - t)N = 0$. We spare the reader the straightforward computations of the identities $(\mathrm{id} - t)b' = b(\mathrm{id} - t)$, $b'N = Nb$, and $s_{-1}b' + b's_{-1} = \mathrm{id}$ (see [**Lod92**, 1.1.12 and 2.1.1]). Finally, we define $B_n\colon X_n \to X_{n+1}$ by $B_n = (\mathrm{id} - t_{n+1})s_{-1}N_n$. From the previous formulas we obtain immediately that $B^2 = 0$ and $bB + Bb = 0$.

By analogy with the Tor definition of Hochschild homology, we define the *cyclic homology* of a cyclic k-module E to be the graded k-module

$$HC_n(E) = \mathrm{Tor}_n^{\Lambda}(E,k).$$

We prove that this is well-defined by producing a projective resolution of the constant cocyclic k-module k. It will be constructed as the total complex of a double complex K_{**} of cocyclic k-modules. Let $K_{pq} = k[\Lambda_{q-p}] = k[\Lambda(q-p,?)]$ if $q \geq p \geq 0$ and let $K_{pq} = 0$ otherwise. Each column K_{p*} obviously has a structure of cyclic cocyclic k-module, and we take for the vertical (downward) differentials b the ones associated to the underlying simplicial object. The horizontal (leftward) differentials $B_n\colon k[\Lambda_n] \to k[\Lambda_{n+1}]$ are $(1 - t_{n+1})s_{-1}N_n$. The complex K_{**} looks like this:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k[\Lambda_3] & \leftarrow & k[\Lambda_2] & \leftarrow & k[\Lambda_1] & \leftarrow & k[\Lambda_0] \\ \downarrow & & \downarrow & & \downarrow & & \\ k[\Lambda_2] & \leftarrow & k[\Lambda_1] & \leftarrow & k[\Lambda_0] & & \\ \downarrow & & \downarrow & & & & \\ k[\Lambda_1] & \leftarrow & k[\Lambda_0] & & & & \\ \downarrow & & & & & & \\ k[\Lambda_0]. & & & & & & \end{array} \tag{1}$$

We have explained above that this is indeed a bicomplex (with anticommuting squares). There is an augmentation $K_{00} \to k$ sending Λ_0 to 1.

Theorem 2. $\operatorname{Tot} K_{**} \to k$ *is a projective resolution of the constant cocyclic k-module k.*

Proof. Since K_{pq} is a representable functor on $k[\Lambda]$, it is projective by the Mod_k-enriched Yoneda lemma (cf. the proof of theorem 1), and so is any finite product of the K_{pq}'s.

Let us prove that $\operatorname{Tot} K_{**} \to k$ is a resolution of k. Here we may safely forget the cocyclic structures and view $\operatorname{Tot} K_{**}$ and k as chain complexes of cosimplicial k-modules. We introduce an auxiliary bicomplex L_{**} of cosimplicial k-modules:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
k[\Lambda_3] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_3] & \xleftarrow{N} & k[\Lambda_3] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_3] & \leftarrow & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
k[\Lambda_2] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_2] & \xleftarrow{N} & k[\Lambda_2] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_2] & \leftarrow & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
k[\Lambda_1] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_1] & \xleftarrow{N} & k[\Lambda_1] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_1] & \leftarrow & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
k[\Lambda_0] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_0] & \xleftarrow{N} & k[\Lambda_0] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_0] & \leftarrow & \cdots,
\end{array} \tag{2}$$

with the same augmentation $\operatorname{Tot} L_{**} \to k$ as $\operatorname{Tot} K_{**}$. We let M_{**} be the bicomplex obtained from L_{**} by annihilating the even-numbered columns. Let $\phi\colon \operatorname{Tot} K_{**} \to \operatorname{Tot} L_{**}$ be the map induced by $(\mathrm{id}, s_{-1}N_r)\colon k[\Lambda_r] \to k[\Lambda_r] \oplus k[\Lambda_{r+1}]$ and let $\psi\colon \operatorname{Tot} L_{**} \to \operatorname{Tot} M_{**}$ be the one induced by $-s_{-1}N_r + \mathrm{id}\colon k[\Lambda_r] \oplus k[\Lambda_{r+1}] \to k[\Lambda_{r+1}]$. The proof that ϕ and ψ are chain maps uses only the relations between b, b', $\mathrm{id} - t$, N, and B that we already wrote down. Moreover, ϕ is compatible with the augmentations to k as ϕ_0 is the identity. Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Tot} K_{**} & \xrightarrow{\phi} & \operatorname{Tot} L_{**} & \xrightarrow{\psi} & \operatorname{Tot} M_{**} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k & = & k & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{3}$$

whose rows are obviously exact. To complete the proof we will show that the last two vertical arrows are quasi-isomorphisms: it will then follow from the associated long exact sequences and the five lemma that the first vertical arrow is also a quasi-isomorphism. From the identity $s_{-1}b' + b's_{-1} = \mathrm{id}$ we obtain that each column of M_{**} has zero homology, and hence that that the third vertical map in (3) is a quasi-isomorphism. Next we show that each row in (2) has zero positive homology, so that the homology of $\operatorname{Tot} L_{**}$ can be computed as the homology of the zeroth column of horizontal homology of L_{**}.† This can be proved pointwise, so consider a part of the nth row evaluated at m:

$$\cdots \to k[\Lambda(n,m)] \xrightarrow{\mathrm{id}-t} k[\Lambda(n,m)] \xrightarrow{N} k[\Lambda(n,m)] \to \cdots . \tag{4}$$

By the structure theorem for Λ, we have $\Lambda(n,m) = \mathbb{Z}_{n+1} \times \Delta(n,m)$, where $\mathbb{Z}_{n+1}$ is the set of automorphisms of n in Λ, and we see that (4) is obtained from a complex of k-modules

$$\cdots \to k[\mathbb{Z}_{n+1}] \xrightarrow{\mathrm{id}-t} k[\mathbb{Z}_{n+1}] \xrightarrow{N} k[\mathbb{Z}_{n+1}] \to \cdots \tag{5}$$

by applying the exact functor $M \mapsto M^{\Delta(n,m)}$, so we need only prove that (5) is exact. Let $x = (x_0, \dots, x_n) \in k[\mathbb{Z}_{n+1}]$. Suppose first that $(\mathrm{id} - t_n)x = 0$; then we obtain successively $x_0 = (-1)^n x_1 = \cdots = (-1)^{nn} x_n$ and hence $x = N(x_0, 0 \dots, 0)$. Suppose then that $Nx = 0$, i.e. that $\sum_{i=0}^{n} (-1)^{ni} x_i = 0$; then putting $y_0 = 0$, $y_1 = -x_0$, $y_2 = -x_0 - x_1$, $\dots$, $y_n = -\sum_{i=0}^{n-1} x_i$, we

†The proof of this fact given here is from [**Con83**]. We should note that the shorter proof in [**Lod92**] is incorrect.

find $(\mathrm{id} - t_n)y = x$. These calculations also show that the image of $\mathrm{id} - t_n\colon k[\mathbb{Z}_{n+1}] \to k[\mathbb{Z}_{n+1}]$ is exactly the kernel of the surjective map $k[\mathbb{Z}_{n+1}] \to k$, $(x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n}(-1)^{ni}x_i$, so that the homology of the nth row in degree 0 can be identified with $k[\Delta_n]$. Moreover, the map $k[\Delta_n] \to k[\Delta_{n-1}]$ induced by b is exactly the differential of the chain complex $k[\Delta_*]$ considered in the proof of Theorem 1. Since we know that the latter is a resolution of the constant cosimplicial k-module k, this proves that $\operatorname{Tot} L_{**} \to k$ is a resolution of k, i.e., that the second vertical arrow in (3) is a quasi-isomorphism. □

Let E be a cyclic k-module. We let $\mathscr{B}(E)$ denote the bicomplex of k-modules obtained by applying the functor $E \otimes_{\Lambda} ?$ to (1). Using the same maps α and β as in the proof of theorem 1, we obtain a natural isomorphism of simplicial k-modules $E \otimes_{\Lambda} K_{p,*+p} \cong E_*$, and so the nth column of $\mathscr{B}(E)$ is just the complex associated to E with E_0 in degree n. We still write B_n for the horizontal differential $\mathrm{id}_E \otimes_{\Lambda} B_n$, which is identified to $\alpha_{n+1}(\mathrm{id}_E \otimes_{\Lambda} B_n)\beta_n\colon E_n \to E_{n+1}$. Explicitly, we have $B_n = (1 - t_{n+1})s_{-1}N_n$.

By definition, the cyclic homology of E is the total homology of $\mathscr{B}(E)$. The bicomplex $\mathscr{B}(E)$ has an obvious periodic pattern, and it is useful to fill it on the left to obtain the bicomplex $\mathscr{B}^{\mathrm{per}}(E)$:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \leftarrow & E_3 & \leftarrow & E_2 & \leftarrow & E_1 & \leftarrow & E_0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \leftarrow & E_2 & \leftarrow & E_1 & \leftarrow & E_0 & & \\
 & & \downarrow & & \downarrow & & & & \\
\cdots & \leftarrow & E_1 & \leftarrow & E_0 & & & & \\
 & & \downarrow & & & & & & \\
\cdots & \leftarrow & E_0 & & & & & &
\end{array}
$$

The complex $\mathscr{B}(E)$ is obtained from $\mathscr{B}^{\mathrm{per}}(E)$ by removing the negatively graded columns. If one removes the positively graded columns instead, one obtains a bicomplex $\mathscr{B}^{-}(E)$. In figurative terms, the bicomplexes $\mathscr{B}^{-}(E)$ and $\mathscr{B}(E)$ "cover" $\mathscr{B}^{\mathrm{per}}(E)$ and their "intersection" is just the 0th column, which we denote by $\mathscr{B}^{0}(E)$. Then the total homology of $\mathscr{B}^{\mathrm{per}}(E)$ (resp. of $\mathscr{B}^{-}(E)$) is called the *periodic cyclic homology* (resp. the *negative cyclic homology*) of E, and it is denoted by $HC^{\mathrm{per}}(E)$ (resp. by $HC^{-}(E)$).

If C is a bicomplex, $C[m,n]$ will denote the bicomplex with $C[m,n]_{pq} = C_{p+m,q+n}$; one has $(\operatorname{Tot} C)_{k+m+n} = (\operatorname{Tot} C[m,n])_k$. The periodicity of $\mathscr{B}^{\mathrm{per}}(E)$ is then expressed by $\mathscr{B}^{\mathrm{per}}(E) = \mathscr{B}^{\mathrm{per}}(E)[1,1]$. It is obvious that there are diagrams of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \mathscr{B}^{-}[1,1] & \to & \mathscr{B}^{-} & \longrightarrow & \mathscr{B}^{0} & \longrightarrow & 0 \\
 & & \| & & \hookdownarrow\twoheaduparrow & & \hookdownarrow\twoheaduparrow & & \\
0 & \to & \mathscr{B}^{-}[1,1] & \to & \mathscr{B}^{\mathrm{per}} & \longrightarrow & \mathscr{B} & \longrightarrow & 0 \\
 & & \hookdownarrow & & \| & & \twoheaddownarrow & & \\
0 & \longrightarrow & \mathscr{B}^{-} & \longrightarrow & \mathscr{B}^{\mathrm{per}} & \to & \mathscr{B}[-1,-1] & \to & 0 \\
 & & \twoheaddownarrow\hookuparrow & & \twoheaddownarrow\hookuparrow & & \| & & \\
0 & \longrightarrow & \mathscr{B}^{0} & \longrightarrow & \mathscr{B} & \longrightarrow & \mathscr{B}[-1,-1] & \to & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \leftarrow & \mathscr{B}^{-}[1,1] & \leftarrow & \mathscr{B}^{-} & \longleftarrow & \mathscr{B}^{0} & \longleftarrow & 0 \\
 & & \| & & \hookdownarrow\twoheaduparrow & & \hookdownarrow\twoheaduparrow & & \\
0 & \leftarrow & \mathscr{B}^{-}[1,1] & \leftarrow & \mathscr{B}^{\mathrm{per}} & \longleftarrow & \mathscr{B} & \longleftarrow & 0 \\
 & & \twoheaduparrow & & \| & & \hookuparrow & & \\
0 & \longleftarrow & \mathscr{B}^{-} & \longleftarrow & \mathscr{B}^{\mathrm{per}} & \leftarrow & \mathscr{B}[-1,-1] & \leftarrow & 0 \\
 & & \twoheaddownarrow\hookuparrow & & \twoheaddownarrow\hookuparrow & & \| & & \\
0 & \longleftarrow & \mathscr{B}^{0} & \longleftarrow & \mathscr{B} & \longleftarrow & \mathscr{B}[-1,-1] & \leftarrow & 0
\end{array}
$$

where pairs of arrows are retract pairs. From these two diagrams we obtain eight functorial long exact sequences and morphisms between them.

Corollary 3. *Let E and F be cyclic k-modules and let $f, g\colon E \rightrightarrows F$ be maps of cyclic k-modules. Then $HH(f) = HH(g)$ if and only if $HC(f) = HC(g)$. When this is the case $HC^{\mathrm{per}}(f) = HC^{\mathrm{per}}(g)$ and $HC^{-}(f) = HC^{-}(g)$.*

Proof. The first statement follows from an inductive five-lemma analysis of the long exact sequences

$$\begin{array}{ccccccccccc}\cdots\to HC_{n-1}(E) & \to & HH_n(E) & \to & HC_n(E) & \to & HC_{n-2}(E) & \to & HH_{n-1}(E) & \to\cdots\\ \downdownarrows && \downdownarrows && \downdownarrows && \downdownarrows && \downdownarrows & \\ \cdots\to HC_{n-1}(F) & \to & HH_n(F) & \to & HC_n(F) & \to & HC_{n-2}(F) & \to & HH_{n-1}(F) & \to\cdots\end{array}$$

$$\begin{array}{ccccccccccccc}\cdots\to HC_0(E) & \to & HH_1(E) & \to & HC_1(E) & \to & 0 & \to & HH_0(E) & \to & HC_0(E) & \to & 0\\ \downdownarrows && \downdownarrows && \downdownarrows && \downdownarrows && \downdownarrows && \downdownarrows && \\ \cdots\to HC_0(F) & \to & HH_1(F) & \to & HC_1(F) & \to & 0 & \to & HH_0(F) & \to & HC_0(F) & \to & 0\end{array}$$

induced by the last line of the diagram on the left above, where the vertical maps are induced by f and g. The other results come from the fact that maps of bicomplexes that are homologically equal on each column are globally homologically equal. This is proved using either a staircase argument or a spectral sequence argument. □

Of course, if A is a k-algebra, $HC(A)$, $HC^{\mathrm{per}}(A)$, and $HC^-(A)$ stand for $HC(C(A))$, $HC^{\mathrm{per}}(C(A))$, and $HC^-(C(A))$.

1.2 Morita invariance

Let A and B be k-algebras. A well-known theorem of commutative algebra (see e.g. [**Bas68**, Thm. 2.3]) says that there is a bijection between isomorphism classes of (B, A)-bimodules and isomorphism classes of colimit-preserving k-linear functors $\mathsf{Mod}_A \to \mathsf{Mod}_B$ (here Mod_A denotes the category of *left* A-modules). This correspondence is given explicitly as follows: to a (B, A)-bimodule ${}_BM_A$ we associate the functor ${}_BM_A\otimes ?$, and to a k-linear functor $F\colon \mathsf{Mod}_A \to \mathsf{Mod}_B$ we associate the (B, A)-bimodule $F(A)$ whose right A-module structure is given by the composition

$$A \xrightarrow{\cong} \mathrm{Hom}_A(A, A) \xrightarrow{F} \mathrm{Hom}_B(F(A), F(A)),$$

where the first map is the action of A on itself by multiplication on the right. This bijection transforms tensor products of bimodules into compositions of functors.[†] In particular, the categories Mod_A and Mod_B are k-linearly equivalent if and only if there exist bimodules ${}_BP_A$ and ${}_AQ_B$ and isomorphisms ${}_AQ_B \otimes {}_BP_A \cong A$ and ${}_BP_A \otimes {}_AQ_B \cong B$ of A-bimodules and B-bimodules, respectively; if F and G are mutually inverse equivalences, we can choose $P = F(A)$ and $Q = G(B)$. When either of those two conditions is satisfied, we say that A and B are *Morita equivalent*, and P, Q, F, and G are called *Morita equivalences.*

Let us quickly review the main results of Morita theory (proofs can be found in [**Bas68**] or [**Lam98**]). Let A be a k-algebra. A left A-module P is automatically an $(A, \mathrm{End}_A(P)^{\mathrm{op}})$-bimodule, and its dual $P^* = \mathrm{Hom}_A(P, A)$ is an $(\mathrm{End}_A(P)^{\mathrm{op}}, A)$-bimodule. There is a canonical map of $\mathrm{End}_A(P)^{\mathrm{op}}$-bimodules

$$P^* \otimes_A P \to \mathrm{End}_A(P)$$

given by $\xi \otimes x \mapsto \xi(?)x$ and a canonical map of A-bimodules

$$P \otimes_{\mathrm{End}_A(P)^{\mathrm{op}}} P^* \to A$$

given by $x \otimes \xi \mapsto \xi(x)$. It is well-known and easy to prove that the first map is an isomorphism if and only if P is a finitely generated projective module. It can also be proved, but we will never use it, that the the second map is an isomorphism precisely when P is a generator of Mod_A in the categorical sense, i.e., when the functor $\mathsf{Mod}_A(P, ?)$ is faithful; we then say that P is *generating.*

The Morita theorems say that P is an f.g.p. and generating left A-module if and only if the categories Mod_A and $\mathsf{Mod}_{\mathrm{End}_A(P)^{\mathrm{op}}}$ are equivalent; $P^* \otimes_A ?$ and $P \otimes_{\mathrm{End}_A(P)^{\mathrm{op}}} ?$ are then quasi-inverse equivalences. Moreover, all Morita equivalences are of this form, in the following sense: if A and B are k-algebras, then a (B, A)-bimodule ${}_BP_A$ is a Morita equivalence if and

[†] Other formal properties of this bijection could be succintly summarized by proving that it comes from an equivalence of suitable 2-categories.

only if it is generating and f.g.p. as a left B-module and the action of A induces an isomorphism $A \to \mathrm{End}_B(P)^{\mathrm{op}}$. When this is the case, $B \to \mathrm{End}_A(P)$ is also an isomorphism of k-algebras, and $\mathrm{Hom}_A(P, A)$ and $\mathrm{Hom}_B(P, B)$ are isomorphic as (A, B)-bimodules and are inverse to ${}_BP_A$.

In this section we shall prove that the functors K_0, HH, HC, HC^-, and HC^{per} send Morita equivalent k-algebras to isomorphic objects. But we want to express this fact more functorially so that the Chern character becomes "Morita natural". The obvious category to consider is that with k-algebras as objects and isomorphism classes of bimodules as morphisms between them. However, the functor K_0 does not lift to this category because tensoring with arbitrary bimodules does not preserve f.g.p. modules. This motivates the following definition.

Define the category Mor_k as follows. Its objects are (associative and unital) k-algebras, and a morphism from A to B is the isomorphism class of a (B, A)-bimodule ${}_BM_A$ that is f.g.p. as a left B-module. The composition of two morphisms ${}_BM_A$ and ${}_CN_B$ is the (C, A)-bimodule ${}_CN_B \otimes {}_BM_A$, and the identity morphism at A is ${}_AA_A$. There is a functor $\mathsf{Alg}_k \to \mathsf{Mor}_k$ which sends a map $f \colon A \to B$ to the (B, A)-bimodule ${}_BB_A$ whose right A-module structure is induced by f. This functor is faithful, for if f and g have the same image, then for all $a \in A$ and $b \in B$, $bf(a) = bg(a)$; taking $b = 1$, we get $f = g$.

Let F be any functor defined on Alg_k. We say that F is *Morita invariant* if it factors through the functor $\mathsf{Alg}_k \to \mathsf{Mor}_k$ defined above. This implies in particular that F sends Morita equivalent k-algebras to isomorphic objects. If F and G are two Morita invariant functors with given factorizations, then we have two notions of a natural transformation $F \to G$. When F and G are viewed as functors on Mor_k a natural transformation $F \to G$ will be called *Morita natural*. Any Morita natural transformation is in particular a natural transformation between functors on Alg_k.

We shall see that the functors that interest us are all Morita invariant in this sense. Here is an easy example of a Morita invariant functor, which motivated our definition of Mor_k.

Theorem 4. *The functors μ and K_0 are Morita invariant.*

Proof. It suffices to prove the theorem for μ. Let ${}_BP_A$ be a morphism from A to B in Mor_k. It induces a colimit-preserving functor between the categories of f.g.p. left A-modules and f.g.p. left B-modules, whence a monoid morphism $\mu({}_BP_A) \colon \mu(A) \to \mu(B)$. When ${}_BP_A$ is the image of a map $f \colon A \to B$, this is exactly how we defined $\mu(f)$. The fact that this defines a functor on Mor_k is obvious. ☐

We continue to write μ and K_0 for the extensions of these functors to Mor_k. Observe that μ is just the functor represented by k on Mor_k, and similarly K_0 is the functor represented by k on the category obtained from Mor_k by turning the monoids of morphisms into groups. Note however that if ${}_BP_A$ is a bimodule where A and B are commutative algebras, then $K_0({}_BP_A)$ is not a morphism of rings in general.

Less trivial is the fact that Hochschild homology and the various cyclic homologies are Morita invariant. The construction of their lift that we present here is slightly simplified from that in Loday [**Lod92**].

Let A and B be k-algebras and let ${}_BP_A$ be a morphism in Mor_k. Write P^* for $\mathrm{Hom}_B(P, B)$. Then P^* is an $(\mathrm{End}_B(P)^{\mathrm{op}}, B)$-bimodule, and also an (A, B)-bimodule thanks to the map of k-algebras $A \to \mathrm{End}_B(P)^{\mathrm{op}}$. Recall from the beginning of this section that we have canonical maps

$$u \colon P^* \otimes_B P \to \mathrm{End}_B(P), \quad \pi \otimes p \mapsto \pi(?)p,$$

and

$$v \colon P \otimes_{\mathrm{End}_B(P)^{\mathrm{op}}} P^* \to B, \quad p \otimes \pi \mapsto \pi(p),$$

of $\mathrm{End}_B(P)^{\mathrm{op}}$-bimodules and B-bimodules, respectively. We think of these two maps as products and we write simply πp for $u(\pi \otimes p)$, $p\pi$ for $v(p \otimes \pi)$, $\pi b p$ for $u(\pi b \otimes p) = u(\pi \otimes bp)$, and $p\alpha\pi$ for $v(p\alpha \otimes \pi) = v(p \otimes \alpha\pi)$. These products are then associative: we have $(\pi p)\pi' = \pi(p\pi')$ and $(p\pi)p' = p(\pi p')$, as is clear from the definitions of u and v. Other associativity identities follow formally using the linearity of u and v, such as $(p\pi)(p'\pi') = p(\pi p')\pi'$. In these notations we also identify elements of A with their image in $\mathrm{End}_B(P)^{\mathrm{op}}$. For example, if $a \in A$, $pa\pi$ stands for $v(pa \otimes \pi) = v(p \otimes a\pi) = \pi(pa)$. This turns out to be a very effective formalism.

By definition of the morphisms in Mor_k, u is an isomorphism. Thus there is a *canonical element* $\pi_j \otimes p^j$ of $P^* \otimes_B P$ such that $\pi_j p^j = \mathrm{id}_P$ (here and in the sequel we use Einstein's sum convention). Let N be any B-bimodule and let ${}_AM_A = P^* \otimes_B N \otimes_B P$. We then define a k-module map $\phi_n\colon C_n(A,M) \to C_n(B,N)$ by the formula

$$\phi_n((\pi \otimes y \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = (p^{j_0}\pi)y(p\pi_{j_1}) \otimes p^{j_1}a_1\pi_{j_2} \otimes \cdots \otimes p^{j_n}a_n\pi_{j_0}. \qquad (6)$$

We omit the easy proof that if $k\colon P \to P'$ is an isomorphism of (B,A)-bimodules, then the map ϕ'_n defined from P' is equal to ϕ_n, as long as we use the decomposition $(k^*)^{-1}(\pi_j) \otimes k(p^j)$ of the canonical element of $P'^* \otimes_B P'$.

Lemma 5. *The map ϕ is a morphism of simplicial k-modules. If we are given different decompositions of the canonical element of $P^* \otimes_B P$, the resulting maps are simplicially homotopic. In particular, the map induced by ϕ on Hochschild homology depends only on P.*

Proof. For the first statement we only verify that $d_i\phi_n = \phi_{n-1}d_i$ for $1 \le i \le n-1$, but the method is the same for the other cases. We have

$$d_i\phi_n((\pi \otimes y \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = (p^{j_0}\pi)y(p\pi_{j_1}) \otimes p^{j_1}a_1\pi_{j_2} \otimes \cdots \otimes (p^{j_i}a_i\pi_{j_{i+1}})(p^{j_{i+1}}a_{i+1}\pi_{j_{i+2}}) \otimes \cdots \otimes p^{j_n}a_n\pi_{j_0}$$

and

$$\phi_{n-1}d_i((\pi \otimes y \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = (p^{j_0}\pi)y(p\pi_{j_1}) \otimes p^{j_1}a_1\pi_{j_2} \otimes \cdots \otimes p^{j_i}(a_ia_{i+1})\pi_{j_{i+2}} \otimes \cdots \otimes p^{j_n}a_n\pi_{j_0},$$

where we have judiciously indexed the indices in the second expression. To see that the two expressions are equal, just insert $\mathrm{id}_P = \pi_{j_{i+1}}p^{j_{i+1}}$ between a_i and a_{i+1} in the second expression and use associativity.

For the second statement, suppose that we are given another decomposition $\kappa_k \otimes q^k$ of the canonical element for P, and let ψ be the map defined from it. For $0 \le i \le n$, define a map $h_i\colon C_n(A,M) \to C_{n+1}(B,N)$ by

$$h_i((\pi \otimes y \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = (p^{j_0}\pi)y(p\kappa_{k_1}) \otimes q^{k_1}a_1\kappa_{k_2} \otimes \cdots \otimes q^{k_i}a_i\kappa_{k_{i+1}} \otimes q^{k_{i+1}}\pi_{j_{i+1}} \otimes p^{j_{i+1}}a_{i+1}\pi_{j_{i+2}} \otimes \cdots \otimes p^{j_n}a_n\pi_{j_0}.$$

The identites for simplicial homotopies that we must check are $d_0h_0 = \phi_n$, $d_{n+1}h_n = \psi_n$, $d_ih_j = h_{j-1}d_i$ for $i < j$, $d_ih_i = d_ih_{i-1}$, and $d_ih_j = h_jd_{i-1}$ for $i > j+1$. All are seen to hold without writing anything down, keeping in mind that $\pi_jp^j = \kappa_kq^k = \mathrm{id}_P$. □

Lemma 6. *Let A, B, C be k-algebras, ${}_BP_A$ and ${}_CQ_B$ morphisms in Mor_k, L a C-bimodule, ${}_BN_B = Q^* \otimes_C L \otimes_C Q$, and ${}_AM_A = P^* \otimes_B N \otimes_B P$. Define as above maps of simplicial k-modules $\phi\colon C(A,M) \to C(B,N)$, $\psi\colon C(B,N) \to C(C,L)$, and $\chi\colon C(A,M) \to C(C,L)$ from P, Q, and $Q \otimes_B P$, respectively. Then $\psi\phi$ and χ are simplicially homotopic.*

Proof. Suppose that ϕ and ψ were defined from the decompositions $\pi_j \otimes p^j$ and $\kappa_k \otimes q^k$ of the canonical elements for P and Q. In the statement of the lemma, we have tacitly used the canonical morphism of (A,C)-bimodules $P^* \otimes_B Q^* \to (Q \otimes_B P)^*$ to define χ. This morphism sends $\pi \otimes \kappa$ to the left C-linear form $q \otimes p \mapsto q(p\pi)\kappa$, and because P and Q are left f.g.p. it is actually an isomorphism whose inverse sends an element $\zeta \in (Q \otimes_B P)^*$ to $\pi_j \otimes \kappa_k\zeta(q^k \otimes p^j)$. We shall write $\pi \otimes \kappa$ for either an element of $P^* \otimes_B Q^*$ or of $(Q \otimes_B P)^*$.

Now $(\pi_j \otimes \kappa_k) \otimes (q^k \otimes p^j)$ is the canonical element for $Q \otimes_B P$, because for any $q \otimes p \in Q \otimes_B P$,

$$(q \otimes p)(\pi_j \otimes \kappa_k)(q^k \otimes p^j) = q(p\pi_j)\kappa_kq^k \otimes p^j = q(p\pi_j) \otimes p^j = q \otimes p\pi_jp^j = q \otimes p.$$

By lemma 5, the map $C(A,M) \to C(C,L)$ obtained from this decomposition is simplicially homotopic to χ, so we shall assume that it is χ and we shall prove $\psi\phi = \chi$. On the one hand

$$\psi_n\phi_n((\pi \otimes \kappa \otimes z \otimes q \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = (q^{k_0}(p^{j_0}\pi)\kappa)z(q(p\pi_{j_1})\kappa_{k_1}) \otimes q^{k_1}(p^{j_1}a_1\pi_{j_2})\kappa_{k_2} \otimes \cdots \otimes q^{k_n}(p^{j_n}a_n\pi_{j_0})\kappa_{k_0},$$

and on the other hand

$$\chi_n((\pi \otimes \kappa \otimes z \otimes q \otimes p) \otimes a_1 \otimes \cdots \otimes a_n) = \\ ((q^{k_0} \otimes p^{j_0})(\pi \otimes \kappa))z((q \otimes p)(\pi_{j_1} \otimes \kappa_{k_1})) \otimes (q^{k_1} \otimes p^{j_1})a_1(\pi_{j_2} \otimes \kappa_{k_2}) \\ \otimes \cdots \otimes (q^{k_n} \otimes p^{j_n})a_n(\pi_{j_0} \otimes \kappa_{k_0}).$$

These expressions are seen to be equal by inspection. □

Lemma 7. *Let A and B be k-algebras, ${}_BP_A$ a morphism in Mor_k, and $N \to N'$ a morphism of B-bimodules. Then the square*

$$\begin{CD} C(A, P^* \otimes_B N \otimes_B P) @>>> C(B, N) \\ @VVV @VVV \\ C(A, P^* \otimes_B N' \otimes_B P) @>>> C(B, N') \end{CD}$$

is commutative, provided that the same decomposition of the canonical element of $P^* \otimes_B P$ is used to define both horizontal maps.

Proof. This is obvious from (6). □

Finally, we note that if ${}_BB_A$ is the image of a map of k-algebras $f\colon A \to B$ and if N is any B-bimodule, then the map $C(A, f^*N) \to C(B, N)$ induced by ${}_BB_A$ is simplicially homotopic to $C(f, \mathrm{id})$. In fact, they are equal if one uses the decomposition $\mathrm{id}_B \otimes 1 \in B^* \otimes_B B$ of the canonical element, for then in the formula (6) the factors are of the form $1a_i$, which is $f(a_i)$ by definition of the action of A on B.

Let us gather the consequences of these results for the special case where the B-bimodule N is just B (where A, B, P, ϕ are as before). We define a map $C(A) \to C(B)$ as the composition

$$C(A, A) \to C(A, P^* \otimes_B P) \xrightarrow{\phi} C(B, B) \tag{7}$$

where the first map is induced by the map of A-bimodules

$$A \to \mathrm{End}_B(P) \xrightarrow{u^{-1}} P^* \otimes_B P\ .$$

If C is a third k-algebra and ${}_CQ_B$ is a morphism in Mor_k, we can form the diagram

$$\begin{array}{ccccc} C(A, A) & \longrightarrow & C(A, P^* \otimes_B P) & \longrightarrow & C(B, B) \\ & \searrow & \downarrow & & \downarrow \\ & & C(A, P^* \otimes_B Q^* \otimes_C Q \otimes_B P) & \to & C(B, Q^* \otimes_C Q) \\ & & & \searrow & \downarrow \\ & & & & C(C, C) \end{array}$$

in which all the maps have been defined. The upper left triangle commutes by the classical functoriality of the Hochschild complex; the square commutes by lemma 7; and the lower triangle commutes up to simplicial homotopy by lemma 6.

In case ${}_BP_A = {}_BB_A$ is the image of $f\colon A \to B$, the second map in (7) can be chosen to be $C(f, \mathrm{id})$, as we have just seen above, so by functoriality of the Hochschild complex, (7) is exactly $C(f)$. Thus, the Hochschild complex functor with values in the homotopy category of simplicial k-modules lifts to Mor_k. If now we pass to the homotopy groups, we obtain:

Theorem 8. *The functor HH from k-algebras to graded k-modules is Morita invariant.*

In fact, lemmas 6 and 7 prove more generally that the functor HH lifts from the fibered category of pairs $(A, {}_AM_A)$ to the category with the same objects but in which a morphism from (A, M) to (B, N) is a morphism ${}_BP_A$ of Mor_k together with a map of A-bimodules $M \to P^* \otimes_B N \otimes_B P$. If we wanted to state this fact precisely we would run into problems with the use of isomorphism classes (we would probably need to define Mor_k as a 2-category), so we shall not attempt it.

We continue to write HH for the lift of HH to Mor_k. We can of course give an explicit description of $HH({}_BP_A)\colon HH(A) \to HH(B)$. If $\pi_j \otimes p^j$ is a decomposition of the canonical element, it is induced by the simplicial morphism $\phi\colon C(A) \to C(B)$ given by

$$\phi_n(a_0 \otimes \cdots \otimes a_n) = p^{j_0} a_0 \pi_{j_1} \otimes p^{j_1} a_1 \pi_{j_2} \otimes \cdots \otimes p^{j_n} a_n \pi_{j_0}. \tag{8}$$

We sometimes write $\phi = C({}_BP_A)$ when the fact that it is only well-defined up to simplicial homotopy is irrelevant.

There remains to handle the various cyclic homologies, but this is now easy. It is obvious from the structure of the map ϕ_n above that it commutes with the cyclic operator t_n, so that it is in fact a map of cyclic k-modules. We still have to prove that the map induced by ϕ on cyclic homology does not depend on the choice of a decomposition of the canonical element, and that composition is preserved, but this follows from corollary 3. Thus:

Theorem 9. *HC, HC^-, and HC^{per} are Morita invariant.*

We continue to write HC, HC^-, and HC^{per} for the lifts of these functors to Mor_k.

One can think of the category Mor_k as an "additivization" of Alg_k. Indeed, Mor_k is an additive category: it is enriched over abelian monoids (that one can replace by abelian groups if one wants to, but this is not necessary), it has a zero object, and it has biproducts. The monoid structure on $\mathsf{Mor}_k(A, B)$ is of course induced by the direct sum of bimodules, and composition in Mor_k becomes biadditive. The zero element is the zero algebra 0, because any left or right 0-module is zero (if M is a 0-module and $x \in M$, then $x = 1x = (1+1)x = x+x$ whence $x = 0$). Finally, for any k-algebras A and B there is a biproduct diagram

$$A \underset{{}_{A\times B}A_A}{\overset{{}_AA_{A\times B}}{\leftrightarrows}} A \times B \underset{{}_{A\times B}B_B}{\overset{{}_BB_{A\times B}}{\rightleftarrows}} B.$$

where $A \times B$ acts on A or B via the projections. The identites for biproducts that we must check are

$$\begin{gathered} {}_AA_{A\times B} \otimes {}_{A\times B}A_A \cong {}_AA_A, \\ {}_BB_{A\times B} \otimes {}_{A\times B}B_B \cong {}_BB_B, \text{ and} \\ ({}_{A\times B}A_A \otimes {}_AA_{A\times B}) \oplus ({}_{A\times B}B_B \otimes {}_BB_{A\times B}) \cong {}_{A\times B}(A \times B)_{A\times B}, \end{gathered}$$

all of which are clear. In any category enriched over abelian monoids, a biproduct provides at the same time the product and the coproduct. Since the bimodules ${}_AA_{A\times B}$ and ${}_BB_{A\times B}$ are the images of the projections in Alg_k and since 0 is terminal in Alg_k, we see that the functor $\mathsf{Alg}_k \to \mathsf{Mor}_k$ creates finite products.

The Hochschild complex functor is additive on Mor_k up to simplicial homotopy. To see this, note that if $\pi_j \otimes p^j$ (resp. $\kappa_k \otimes q^k$) is the canonical element for ${}_BP_A$ (resp. ${}_BQ_A$), then $(\pi_j + 0) \otimes (p^j + 0) + (0 + \kappa_k) \otimes (0 + q^k)$ is the canonical element for $P \oplus Q$. From the formula (8) we obtain at once $C(P \oplus Q) = C(P) + C(Q)$. It follows that each of the functors HH, HC, HC^-, and HC^{per} is additive on Mor_k and hence preserves zero objects and biproducts. In particular, any of these functors preserves finite products when viewed as a functor on Alg_k.

1.3 The Chern character in degree zero

We begin with another description of the K_0 of a k-algebra A. Let $\mathrm{Mat}_n(A)$ denote the ring of left A-linear endomorphisms of A^n, and let $\mathrm{Mat}(A) = \varinjlim \mathrm{Mat}_n(A)$ (this is a nonunital k-algebra). Using the canonical basis of A^n, we can, and we will, identify elements of $\mathrm{Mat}_n(A)$

with square matrices that act on the left A-module A^n of row vectors by multiplication on the right. To any idempotent e in $\mathrm{Mat}(A)$ we can associate an element $[e]$ of $\mu(A)$, namely the isomorphism class of the image $\operatorname{Im} e$ of e: if we factor e as

$$A^n \xrightarrow{p} \operatorname{Im} e \overset{i}{\hookrightarrow} A^n,$$

then p is a retraction of i, showing that $\operatorname{Im} e$ is an f.g.p. module. Conversely, if M is an f.g.p. left A-module, there is an integer $n \geq 0$ and a commutative diagram

$$M \xrightarrow[i]{} A^n \xrightarrow[p]{} M; \quad (p \circ i = \mathrm{id}_M)$$

then the composite ip is an idempotent in $\mathrm{Mat}(A)$, and M is in the isomorphism class $[ip]$. Moreover, two idempotents e and e' determine the same isomorphism class of f.g.p. modules if and only if there exists $\alpha \in GL(A)$ such that $e' = \alpha e \alpha^{-1}$. Only the necessity is nontrivial, but if $e \in \mathrm{Mat}_n(A)$, $e' \in \mathrm{Mat}_m(A)$, and if $f\colon \operatorname{Im} e \to \operatorname{Im} e'$ is an isomorphism, then

$$\begin{bmatrix} e' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} h & 1-e' \\ 1-e & g \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g & 1-e \\ 1-e' & h \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h & 1-e' \\ 1-e & g \end{bmatrix} \begin{bmatrix} g & 1-e \\ 1-e' & h \end{bmatrix} = 1,$$

where $g\colon A^n \to A^m$ and $h\colon A^m \to A^n$ denote the extensions of f and f^{-1} by zero. Hence, there is a bijection between $\mu(A)$ and the set of orbits of idempotents in $\mathrm{Mat}(A)$ under the action of $GL(A)$ by conjugation.

We now define the Chern character ch_0 with values in $HH_0(A) = HC_0(A) = A/[A, A]$. If e is an idempotent in $\mathrm{Mat}(A)$, $\mathrm{ch}_0([e])$ is the image of e by the composition

$$\mathrm{Mat}(A) \xrightarrow{\mathrm{tr}} A \to A/[A, A].$$

Let us check that this is well-defined. If $[e] = [e']$, we have seen above that there exists $\alpha \in GL(A)$ such that $e' = \alpha e \alpha^{-1}$, so by linear algebra $\mathrm{tr}(e) - \mathrm{tr}(e')$ belongs to the commutator submodule $[A, A]$. Moreover, ch_0 is a morphism of monoids because the direct sum of $[e]$ and $[e']$ is represented by the idempotent

$$e \oplus e' = \begin{bmatrix} e & 0 \\ 0 & e' \end{bmatrix},$$

and the trace of this idempotent is the sum of the traces of e and e'. Thus, by universality, we obtain a morphism of groups

$$\mathrm{ch}_0\colon K_0(A) \to HC_0(A).$$

We want to prove that the map ch_0 is natural when A varies in Alg_k and even in Mor_k. To do this we have to make explicit the way a left f.g.p. bimodule ${}_BP_A$ acts on idempotents in $\mathrm{Mat}(A)$. Let $\pi_j \otimes p^j$ be the canonical element of $P^* \otimes_B P$ with $1 \leq j \leq r$. For any matrix $x \in \mathrm{Mat}_n(A)$, we write $Px \in \mathrm{Mat}_{rn}(B)$ for the matrix

$$\begin{bmatrix} p^1 x \pi_1 & p^1 x \pi_2 & \dots \\ p^2 x \pi_1 & p^2 x \pi_2 & \dots \\ \vdots & \vdots & \end{bmatrix},$$

where $p^i x \pi_j$ is computed coefficient by coefficient. An easy computation shows that $x \mapsto Px$ is a map of nonunital k-algebras. In particular, if e is an idempotent, so is Pe. We claim that

$${}_BP_A \otimes {}_A \operatorname{Im} e \cong {}_B \operatorname{Im} Pe. \tag{9}$$

Define a left B-linear map $\alpha\colon P \otimes_A A^n \to B^{rn}$ by $\alpha(p \otimes a) = (pa\pi_1, \dots, pa\pi_r)$. If $ae = a$, it is clear that $\alpha(p \otimes a)Pe = \alpha(p \otimes a)$, and so α restricts to a map $\alpha'\colon P \otimes_A \operatorname{Im} e \to \operatorname{Im} Pe$. Define $\beta\colon B^{rn} \to P \otimes_A A^n$ by $\beta(b) = b_{ji} p^j \otimes u^i$, where the vector b is divided in r chunks of length n

and where u^i is the ith vector of the canonical basis of A^n. Observe that $\beta\alpha = \mathrm{id}$. Suppose that $b = bPe$, or equivalently that $b_{ji} = b_{lk}p^l e_i^k \pi_j$. Then

$$\begin{aligned}(b_{ji}p^j \otimes u^i)e = b_{lk}p^l e_i^k \pi_j p^j \otimes e^i = b_{lk}p^l e_i^k \otimes e^i = b_{lk}p^l \otimes e_i^k e^i = b_{lk}p^l \otimes (e^2)^k \\ = b_{lk}p^l \otimes e^k = b_{lk}p^l \otimes e_i^k u^i = b_{lk}p^l e_i^k \otimes u^i = b_{lk}p^l e_i^k \pi_j p^j \otimes u^i = b_{ji}p^j \otimes u^i,\end{aligned}$$

that is, $\beta(b)e = \beta(b)$. Thus, β restricts to a map $\beta' \colon \operatorname{Im} Pe \to P \otimes_A \operatorname{Im} e$. It remains to check that $\alpha'\beta' = \mathrm{id}$, or explicitly that for $b \in \operatorname{Im} Pe$, $b_{ji} = b_{mi}p^m \pi_j$. This equality becomes obvious if we multiply both sides on the right by Pe:

$$(bPe)_{ji} = b_{lk}p^l e_i^k \pi_j = b_{mk}p^m \pi_l p^l e_i^k \pi_j = (\alpha'\beta'(b)Pe)_{ji}.$$

Using equation (9) and the formula (8) for $n = 0$ it is now clear that ch_0 is Morita natural, because the trace of Pe is exactly $p^j \operatorname{tr}(e)\pi_j = C_0({}_BP_A)(\operatorname{tr}(e))$.

[The matrix Px has the following origin. If $\hat{P}$ is the unique lift of P as in

$$\begin{array}{ccc} \mathrm{Mat}_n(A) & \xrightarrow{\cong} & A \\ \downarrow{\scriptstyle \hat{P}} & & \downarrow{\scriptstyle P} \\ \mathrm{Mat}_{rn}(B) & \xrightarrow{\cong} & B, \end{array}$$

then the decomposition of the canonical element for P induces one of the canonical element $\hat{\pi}_k \otimes \hat{p}^k$ for $\hat{P}$, and Px is just $\hat{p}^k x \hat{\pi}_k$.]

We use this description of Pe to prove that, if A is commutative, ch_0 is a morphism of rings. Let $e \in \mathrm{Mat}_n(A)$ and $e' \in \mathrm{Mat}_m(A)$ be idempotents. The tensor product $\operatorname{Im} e \otimes_A \operatorname{Im} e'$ can also be viewed as the image of $\operatorname{Im} e'$ under the map $\mu({}_A\mathrm{Im}\, e_A)$, and so it is represented by the idempotent $(\operatorname{Im} e)e'$. The canonical element of $(\operatorname{Im} e)^* \otimes_A \operatorname{Im} e$ is $\pi_j \otimes e^j$ where π_j is the restriction of the jth projection of A^n and e^j is the jth row of e (i.e., the image by e of the jth vector of the canonical basis). Using this decomposition of the canonical element, the matrix $(\operatorname{Im} e)e'$ becomes the classical tensor product matrix $e \otimes e'$, whose trace is $\operatorname{tr}(e)\operatorname{tr}(e')$.

We are now going to prove that ch_0 factors through HC_0^-. This is actually trivial thanks to our work on Morita naturality. As we have noted in §1.2, the functor μ from Mor_k to abelian monoids is represented by k. Let $F \colon \mathsf{Mor}_k \to \mathsf{Ab}$ be any additive functor. Then, by the Yoneda lemma and the universality of K_0, $\tau \mapsto \tau_k(k)$ is an isomorphism from the group of Morita natural transformations $K_0 \to F$ to $F(k)$ which is natural in F. Since the canonical map $HC_0^- \to HC_0$ is an isomorphism on k (as an explicit computation reveals), it follows that any Morita natural transformation $\tau \colon K_0 \to HC_0$ has a unique Morita natural lift τ^- as in the diagram

$$\begin{array}{ccc} K_0 & \xrightarrow{\tau} & HC_0 \\ & {\scriptstyle \tau^-}\searrow & \uparrow \\ & & HC_0^-. \end{array}$$

Explicitly, if M is an f.g.p. left A-module, then

$$\tau^-({}_AM) = HC_0^-({}_AM_k)(\tau^-({}_kk)),$$

where $\tau^-({}_kk)$ is the preimage of $\tau({}_kk)$ by the isomorphism $HC_0^-(k) \to HC_0(k)$. Applying this to ch_0 we obtain the Chern character $\mathrm{ch}_0^- : K_0 \to HC_0^-$.

Remark 1. It may seem that we have not gained much in passing from ch_0 to ch_0^-, whose construction was almost entirely formal. However, we now have natural maps $\mathrm{ch}_{0,n} \colon K_0 \to HC_{2n}$ for all $n \geq 0$ by composing ch_0^- with the canonical maps $HC_0^- \to HC_0^{\mathrm{per}} = HC_{2n}^{\mathrm{per}}$ and $HC_{2n}^{\mathrm{per}} \to HC_{2n}$. On the category of *commutative* k-algebras, there is also a canonical map of graded k-module-valued functors $HC \to H^{\mathrm{dR}}$, where H^{dR} is the de Rham homology functor. Thus, the Chern character ch_0^- of the proposition induces a map from $K_0(A)$ into the even-valued de Rham homology of A (in fact, the Chern character $\mathrm{ch}_{0,n}$ arose as a generalization of this map to the noncommutative case). This provides the connection between the algebraic version of the Chern character discussed here and the topological version discussed elsewhere. If X is a paracompact space, then it is well-known that there exists an isomorphism $K^0(X) \to K_0(C_{\mathbb{C}}(X))$ where $K^0(X)$ is the Grothendieck ring of finite-dimensional complex vector bundles on X and $C_{\mathbb{C}}(X)$ is the $\mathbb{C}$-algebra of complex-valued continuous functions on X. This isomorphism is constructed by associating to a vector bundle on X its $C_{\mathbb{C}}(X)$-module of sections. When X is a C^∞ manifold, the inclusion $i\colon C^\infty_{\mathbb{C}}(X) \to C_{\mathbb{C}}(X)$ induces an isomorphism $i_*\colon K_0(C^\infty_{\mathbb{C}}(X)) \to K_0(C_{\mathbb{C}}(X))$. A famous theorem of de Rham says that there is an isomorphism

$$H^{\mathrm{dR}}(C^\infty_{\mathbb{C}}(X)) \cong H(X, \mathbb{C}),$$

where the H on the right is singular cohomology. All said, we obtain a map

$$K^0(X) \to H(X, \mathbb{C})$$

when X is a manifold. This can be extended to all paracompact spaces using the fact that any vector bundle on a paracompact space X is the pullback of a vector bundle on a manifold (the universal bundle). In this way, it seems, we recover up to a coefficient the Chern character for paracompact spaces as defined in algebraic topology (e.g. using Chern classes).

Remark 2. All that we have done in this chapter generalizes straightforwardly from commutative k-algebras to schemes over k. We mention without proofs the steps of this generalization. First, the K_0 of a scheme X is defined using the category of $\mathscr{O}_X$-modules that are locally free of finite rank (this is equivalent to being locally f.g.p.); it is now a contravariant functor, and, when precomposed by the contravariant spectrum functor, gives back the covariant K_0 for commutative k-algebras. The Hochschild complex of a scheme is defined in exactly the same way as for k-algebras, using the structural sheaf $\mathscr{O}_X$ instead. Thus, we obtain the Hochschild homology and the various cyclic homologies as sheaves of graded k-modules. To obtain k-modules from the latter, we cannot simply take global sections because this operation would not give an invariant of the Hochschild complex under weak equivalences (e.g. simplicial homotopy equivalences). Instead, we have to apply first the total right derived functor $\mathbf{R}\Gamma$ of global sections before taking homology. There are several categories with weak equivalences in which this derived functor can be defined (simplicial sheaves, nonnegatively differential graded sheaves, unbounded differential graded sheaves), but the results obtained are compatible through the Dold–Kan equivalence or the truncation functor (see 4.1). We can still define the Chern character ch_0 and its lift ch_0^- in the same way, and they are natural transformations between contravariant functors.

In the affine case we have seen that K_0 is also a covariant functor (a contravariant functor on algebras) if we impose some finiteness condition on the morphisms. The same is obviously true for all schemes. The Hochschild homology and cyclic homologies of a scheme X are also covariant functors under the same restriction on morphisms. In the case of k-algebras, for example, a map $f\colon A \to B$ that makes B into an f.g.p. A-module induces a map $C({}_AB_B)\colon C(B) \to C(A)$ between the Hochschild complexes. We can then ask if the Chern character for schemes is a natural morphism between covariant functors as well. This is true for affine schemes by Morita naturality. In the general case, however, we guess that there is a Grothendieck–Riemann–Roch formula instead.

2 Stacks over model categories

In this chapter we summarize the results of [**HAGI**] about stacks on model categories. We should note that there exists a more general notion of stack over $(\infty, 1)$-categories and that the two notions are compatible via simplicial localization. But for convenience we state at once the results in a form suited for their applications later on.

2.1 Mapping spaces in model categories

We recall some facts about mapping spaces in model categories. All of them are proved in [**Hov99**]. Let I be an index category. For any category C with colimits, there is an equivalence between the category C^{I} and the category of adjunctions $\mathsf{Set}^{\mathsf{I}^{\mathrm{op}}} \to \mathsf{C}$. If A is a functor $\mathsf{I} \to \mathsf{C}$, its image by this equivalence is an adjunction with left adjoint written $? \otimes A \colon \mathsf{Set}^{\mathsf{I}^{\mathrm{op}}} \to \mathsf{C}$ and with right adjoint written $\mathsf{C}(A, ?) \colon \mathsf{C} \to \mathsf{Set}^{\mathsf{I}^{\mathrm{op}}}$; for K a functor $\mathsf{I}^{\mathrm{op}} \to \mathsf{Set}$, $K \otimes A$ can be described by the glueing procedure

$$K \otimes A = \int^{i \in \mathsf{I}} \coprod_{x \in K(i)} A(i),$$

while $\mathsf{C}(A, X)(i) = \mathsf{C}(A(i), X)$. When K is the functor represented by $i \in \mathsf{I}$, we have $K \otimes A = A(i)$. For example, when $\mathsf{I} = \Delta$, C is the category of topological spaces, and A is the cosimplicial space such that $A(n)$ is the standard topological n-simplex, then $K \otimes A$ is the geometric realization of K and $\mathsf{C}(A, X)$ is the singular simplicial set of X.

Suppose now that C is a model category. We recall the definition of the Reedy model structure on the category of simplicial objects sC whose equivalences are the pointwise equivalences. Since C has all limits and colimits, one can certainly define the skeleton and coskeleton functors $\mathrm{sk}_n \colon \mathsf{sC} \to \mathsf{sC}$ and $\mathrm{cosk}_n \colon \mathsf{sC} \to \mathsf{sC}$ as the compositions

$$\mathrm{sk}_n = (i_n)_! i_n^* \quad \text{and} \quad \mathrm{cosk}_n = (i_n)_* i_n^*,$$

where $i_n^* \colon \mathsf{sC} \to \mathsf{s}_n\mathsf{C}$ is the truncation at n with left adjoint $(i_n)_!$ and right adjoint $(i_n)_*$ (this is defined for all $n \geq -1$). A morphism $X \to Y$ in sC is a (trivial) fibration (resp. a (trivial) cofibration) for the Reedy structure if and only if the induced maps

$$X_n \to \mathrm{cosk}_{n-1}(X)_n \times_{\mathrm{cosk}_{n-1}(Y)_n} Y_n \quad (\text{resp. } \mathrm{sk}_{n-1}(Y)_n \amalg_{\mathrm{sk}_{n-1}(X)_n} X_n \to Y_n)$$

are (trivial) fibrations (resp. (trivial) cofibrations) for all $n \geq 0$ (see [**Hir03**, 15.3.15] for this characterization of trivial fibrations and trivial cofibrations). We shall make use of the following result.

Proposition 10. *Let C be a model category and endow the category sC with its Reedy model structure. For any $n \geq 0$, the functors $\mathrm{sk}_n \colon \mathsf{sC} \to \mathsf{sC}$ and $\mathrm{cosk}_n \colon \mathsf{sC} \to \mathsf{sC}$ form a Quillen adjunction.*

Proof. We check that cosk_n preserves fibrations and trivial fibrations. For a map $f \colon X \to Y$ in sC we write $\mathrm{M}_m f$ for the induced map

$$X_m \to \mathrm{cosk}_{m-1}(X)_m \times_{\mathrm{cosk}_{m-1}(Y)_m} Y_m.$$

Suppose that $f \colon X \to Y$ is a Reedy (trivial) fibration. Using the isomorphisms $\mathrm{cosk}_p \mathrm{cosk}_q \cong \mathrm{cosk}_q \mathrm{cosk}_p \cong \mathrm{cosk}_p$ if $p \leq q$ and $\mathrm{cosk}_n(?)_m = ?_m$ if $m \leq n$, we obtain that $\mathrm{M}_m \mathrm{cosk}_n(f)$ can be identified with $\mathrm{M}_m f$ if $m \leq n$, in which case it is a (trivial) fibration by hypothesis, and is an isomorphism if $m > n$. Thus $\mathrm{cosk}_n(f)$ is a Reedy (trivial) fibration. □

A *cosimplicial resolution functor* on C is a functor $\Gamma^* \colon \mathsf{C} \to \mathsf{C}^{\Delta}$ together with an isomorphism $\Gamma^0 \to \mathrm{id}_{\mathsf{C}}$ such that for every cofibrant object A of C the adjoint map $\Gamma^*(A) \to A$ is a Reedy cofibrant replacement of the constant cosimplicial object A. One defines dually the notion of simplicial resolution functor. The axioms of a model category (with functorial factorizations) imply that cosimplicial and simplicial resolutions always exist. So fix a cosimplicial resolution functor $\Gamma^*\colon \mathsf{C} \to \mathsf{C}^{\Delta}$ and a simplicial resolution functor $\Gamma_*\colon \mathsf{C} \to \mathsf{C}^{\Delta^{\mathrm{op}}}$ on C. Composing these functors with the equivalences of the previous paragraph, we obtain four bifunctors

$$\begin{gathered}\mathsf{sSet} \times \mathsf{C} \to \mathsf{C},\ (K,X) \mapsto K \otimes X,\\ \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{sSet},\ (X,Y) \mapsto \mathrm{Map}_\ell(X,Y),\\ \mathsf{sSet}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C},\ (K,Y) \mapsto Y^K, \text{ and}\\ \mathsf{C} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}^{\mathrm{op}},\ (X,Y) \mapsto \mathrm{Map}_r(X,Y),\end{gathered}$$

with the property that $? \otimes X$ is left adjoint to $\mathrm{Map}_\ell(X, ?)$ and $Y^?$ is right adjoint to $\mathrm{Map}_r(?, Y)$. It turns out that these four bifunctors preserve sufficiently many weak equivalences to have total derived functors $(K,X) \mapsto K \otimes^{\mathbf{L}} X$, $(X,Y) \mapsto \mathbf{R}\,\mathrm{Map}_\ell(X,Y)$, $(K,Y) \mapsto Y^{\mathbf{R}K}$, and $(X,Y) \mapsto \mathbf{L}\,\mathrm{Map}_r(X,Y)$. These derived functors do not depend on Γ^* and Γ_*, in the sense that different choices of cosimplicial and simplicial resolutions yield naturally isomorphic derived bifunctors. Moreover, $\mathbf{R}\,\mathrm{Map}_\ell$ is actually canonically isomorphic to $(\mathbf{L}\,\mathrm{Map}_r)^{\mathrm{op}}$; we simply denote by $\mathbf{R}\,\mathrm{Map}$ one of these two bifunctor (it is further isomorphic to the simplicial hom-set functor of the simplicial localization of the model category C). Now, although the two adjunctions above are not Quillen adjunctions in general, they are if X is cofibrant and if Y is fibrant. Thus if QX denotes a cofibrant replacement of X, then $? \otimes QX$ and $\mathrm{Map}_\ell(QX, ?)$ form a Quillen adjunction for any X, and hence their derived functors, which are exactly $? \otimes^{\mathbf{L}} X$ and $\mathbf{R}\,\mathrm{Map}_\ell(X, ?)$, are adjoint. Similarly, we obtain an adjunction between $\mathbf{L}\,\mathrm{Map}_r(?, Y)$ and $Y^{\mathbf{R}?}$ for any Y. To summarize, there are isomorphisms

$$[K \otimes^{\mathbf{L}} X, Y] \cong [K, \mathbf{R}\,\mathrm{Map}(X,Y)] \cong [X, Y^{\mathbf{R}K}] \tag{10}$$

which are easily seen to be natural in K, in X, and in Y (one only needs to check that the first one is natural in X and the second one in Y). In particular, we obtain that $K \otimes^{\mathbf{L}} ?$ is left adjoint to $?^{\mathbf{R}K}$. It can be proved that the adjunctions (10) are part of a closed $\mathrm{Ho}\,\mathsf{sSet}$-module structure on the category $\mathrm{Ho}\,\mathsf{C}$, where $\mathrm{Ho}\,\mathsf{sSet}$ is endowed with the monoidal structure given by the direct product. These canonical closed $\mathrm{Ho}\,\mathsf{sSet}$-module structures on the homotopy categories of model categories are moreover functorial for Quillen adjunctions: in fact, if $F\colon \mathsf{C} \to \mathsf{D}$ is a colimit-preserving functor between model categories that also preserves cofibrant objects and cofibrations and equivalences between them, then $\mathbf{L}F\colon \mathrm{Ho}\,\mathsf{C} \to \mathrm{Ho}\,\mathsf{D}$ is the underlying functor of a morphism of left $\mathrm{Ho}\,\mathsf{sSet}$-modules, so that $\mathbf{L}F(K \otimes^{\mathbf{L}} X) \cong K \otimes^{\mathbf{L}} \mathbf{L}F(X)$ (this is only proved in [**Hov99**] when F is left Quillen, but exactly these properties of the functor F are used in the proof). Dualizing the hypotheses we obtain that $\mathbf{R}F(Y^{\mathbf{R}K}) \cong \mathbf{R}F(Y)^{\mathbf{R}K}$.

In the special case that C is a simplicial model category, there are canonical choices for resolution functors induced by the sSet-module structure of C, namely $\Gamma^*(A) = \Delta^* \otimes A$ and $\Gamma_*(A) = A^{\Delta^*}$. By abstract nonsense the bifunctors $K \otimes X$ and Y^K induced by these resolutions coincide with those from the sSet-module structure of C. In particular, $\mathrm{Map}_\ell = (\mathrm{Map}_r)^{\mathrm{op}} = \mathrm{Map}$ are just the simplicial hom's of C and the functors $K \otimes ?$ and $?^K$ are already adjoint at the underived level, and this is in fact a Quillen adjunction.

2.2 Prestacks

Let C be a model category and W its set of weak equivalences. We endow the category $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ of simplicial presheaves on C with the projective model structure for which equivalences and fibrations are defined pointwise (this does not use the model structure of C). By virtue of the adjunction between the constant simpicial set functor $\mathsf{Set} \to \mathsf{sSet}$ and the evaluation at 0 functor $\mathsf{sSet} \to \mathsf{Set}$, a simplicial presheaf on C is the same thing as an sSet-enriched presheaf if we view C as an sSet-enriched category with constant morphism objects. Therefore, by the sSet-enriched Yoneda lemma, there is a fully faithful simplicial functor $h\colon \mathsf{C} \to \mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, $x \mapsto h_x$, and for any simplicial presheaf F there is an isomorphism of simplicial sets

$$F(x) \cong \mathrm{Map}(h_x, F)$$

natural in F and in x. This implies that h_x is cofibrant for any $x \in \mathsf{C}$. Indeed, to show that h_x has the left lifting property with respect to a trivial fibration $F \to G$, it suffices to prove that

$F(x)_0 \to G(x)_0$ is surjective. But, by definition of the projective model structure, $F(x) \to G(x)$ is a trivial fibration in sSet and hence any map $* \to G(x)$ lifts to $* \to F(x)$. Objects of the form h_x will be called *representables.* The model category $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ is in a precise sense the homotopical version of the presheaf category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ and has many analogous properties. One of them is that any object in $\mathrm{Ho}\,\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ is in a canonical way a homotopy colimit of a diagram of representables: see [**Dug01**, §2.6].

Recall that $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ is a proper, cellular, and simplicial model category (see [**Hir03**, 13.1.14 and 12.1.5]). We can therefore consider the left Bousfield localization of $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ along the image $h(W)$ of W by the simplicial Yoneda embedding, which is again a left proper, cellular, and simplicial model category ([**Hir03**, 4.1.1]); it is denoted by $\mathsf{C}^\wedge$ and is called the *model category of prestacks* on C. It has the same underlying category, cofibrations (hence also trivial fibrations), and simplicial structure as $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, while its equivalences are the $h(W)$-local equivalences in $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$. By definition, the identity $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{C}^\wedge$ is an equivalence-preserving left Quillen functor enjoying the following universal property: any left Quillen functor $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{D}$ whose total derived functor sends elements of $h(W)$ to isomorphisms in $\mathrm{Ho}\,\mathsf{D}$ lifts uniquely to a left Quillen functor $\mathsf{C}^\wedge \to \mathsf{D}$. Its derived right adjoint $\mathbf{R}\mathrm{id}\colon \mathrm{Ho}\,\mathsf{C}^\wedge \to \mathrm{Ho}\,\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ is simply the functor induced by a fibrant replacement functor in $\mathsf{C}^\wedge$, and it is fully faithful by [**Hir03**, 3.5.1 (1)].[†]

By [**Hir03**, 3.4.1 (1)], an object $F \in \mathsf{C}^\wedge$ is fibrant if and only if it is $h(W)$-local in $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, i.e., if and only if it is pointwise fibrant and for any equivalence $y \to z$ in C, $\mathbf{R}\,\mathrm{Map}(h_z, F) \to \mathbf{R}\,\mathrm{Map}(h_y, F)$ (mapping spaces in $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$) is an isomorphism in $\mathrm{Ho}\,\mathsf{sSet}$. Since h_z and h_y are projectively cofibrant and F is projectively fibrant, those mapping spaces can here be chosen to be the simplicial hom's of $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$. Then the simplicial Yoneda lemma gives us the following criterion: a functor $F\colon \mathsf{C} \to \mathsf{sSet}$ is fibrant in $\mathsf{C}^\wedge$ if and only if

- it is pointwise fibrant and
- it preserves equivalences.

This implies that the essential image of $\mathbf{R}\mathrm{id}\colon \mathrm{Ho}\,\mathsf{C}^\wedge \to \mathrm{Ho}\,\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ consists of the equivalence-preserving functors; such functors are called *prestacks.*

Observe that the simplicial Yoneda embedding $h\colon \mathsf{C} \to \mathsf{C}^\wedge$ preserves weak equivalences by definition, so that it has a total right (and left) derived functor $\mathbf{R}h$. Fix a cosimplicial resolution functor Γ^* on C and a functorial cofibrant replacement $Qx \to x$, and define a functor $\underline{h}\colon \mathsf{C} \to \mathsf{C}^\wedge$ by

$$\underline{h}_x(y) = \mathrm{Map}_\ell(Qy, x) = \mathsf{C}(\Gamma^*(Qy), x).$$

Here we take a cofibrant replacement of y so that $\Gamma^*(Qy) \to Qy$ is a cosimplicial resolution of Qy in the sense of [**Hir03**] (which is only guaranteed for cofibrant objects with our definition of cosimplicial resolution functors). If R is a fibrant replacement functor on C, there is a canonical map

$$h \to \underline{h}_{R?} \tag{11}$$

adjoint to $\mathsf{C}(y, x) \to \mathsf{C}(Qy, x) \cong \mathsf{C}(\Gamma^0(Qy), x) \to \mathsf{C}(\Gamma^0(Qy), Rx)$.

Proposition 11. *The functor $\underline{h}\colon \mathsf{C} \to \mathsf{C}^\wedge$ preserves fibrant objects, fibrations between fibrant objects, equivalences between fibrant objects, and all trivial fibrations. In particular, $\underline{h}$ has a total right derived functor $\mathbf{R}\underline{h}\colon \mathrm{Ho}\,\mathsf{C} \to \mathrm{Ho}\,\mathsf{C}^\wedge$ which underlies a morphism of right* $\mathrm{Ho}\,\mathsf{sSet}$*-modules.*

Proof. As a functor to $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, $\underline{h}$ preserves fibrant objects ([**Hir03**, 16.5.3 (1)]), all fibrations and trivial fibrations ([**Hir03**, 16.5.4 (2)]), and weak equivalences between fibrant objects ([**Hir03**, 16.5.5 (2)]). Now take $\underline{h}\colon \mathsf{C} \to \mathsf{C}^\wedge$. To prove that $\underline{h}_x$ is fibrant for x fibrant, we must prove that for any equivalence $y \to z$ in C, $\underline{h}_x(z) \to \underline{h}_x(y)$ is an equivalence in sSet: this is is [**Hir03**, 16.5.5 (1)]. Since fibrant objects in $\mathsf{C}^\wedge$ are $h(W)$-local in $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, it follows from [**Hir03**, 3.3.16 (1)] that $\underline{h}$ also preserves fibrations between fibrant objects. Finally, $\underline{h}$ preserves weak equivalences

[†]The statement of Lemmas 3.5.1 and 3.5.2 in [**Hir03**] should include a hypothesis of left (right) properness for the proofs given there to work. The missing hypothesis is restored in Proposition 3.5.3. Alternatively, one should replace the condition of (co)locality with the stronger condition of being (co)fibrant in the localization; it is this modified statement that we use here.

between fibrant objects and all trivial fibrations since the identity $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{C}^\wedge$ preserves equivalences and trivial fibrations. □

Proposition 12. *The map (11) is an equivalence in* $\mathsf{C}^\wedge$*, so that* $\mathbf{R}h \cong \mathbf{R}\underline{h}$ *and the latter does not depend on the choice of a cosimplicial resolution functor. For any fibrant* $F \in \mathsf{C}^\wedge$ *and any* $x \in \mathsf{C}$*, there is an isomorphism*

$$F(x) \cong \mathbf{R}\,\mathrm{Map}(\mathbf{R}\underline{h}_x, F)$$

in $\mathrm{Ho}\,\mathsf{sSet}$ *which is natural in* F *and in* x*. In particular,* $\mathbf{R}\underline{h}$ *is fully faithful.*

Proof. The first assertion is proved in [**HAGI**, Lem. 4.2.2]. As F is fibrant and h_x is cofibrant in $\mathsf{C}^\wedge$, it implies that $\mathbf{R}\,\mathrm{Map}(\mathbf{R}\underline{h}_x, F) \cong \mathrm{Map}(h_x, F)$ in $\mathrm{Ho}\,\mathsf{sSet}$, so the other statements follow from the simplicial Yoneda lemma. □

We call $\mathbf{R}\underline{h}$ the *derived Yoneda embedding*.

2.3 Model sites and hypercovers

Let C be a model category. A *model topology* on C is defined to be a topology on $\mathrm{Ho}\,\mathsf{C}$. A model category endowed with a model topology is called a *model site*.

In the remaining of this section we will discuss the notion of hypercovers which will be used to formulate the relevant descent condition for stacks in the next section. Hypercovers are a generalization of the classical notion of Čech cover, which we recall first.

Let (C, τ) be a classical site with fibered products. To a covering family $U = \{x_i \to x\}_i$ of an object $x \in \mathsf{C}$ (this means that the sieve generated by U is a covering sieve for τ), one associates an augmented simplicial object $C_*(U) \to x$, called the *nerve* of U, which in degree n is a "formal disjoint union" of intersections

$$\coprod_{i_0,\dots,i_n} x_{i_0} \times_x \cdots \times_x x_{i_n}.$$

The classical descent condition is that a presheaf of sets F on C is a sheaf if and only if for every covering family U the map $F(x) \to \varprojlim F(C_*(U))$ is an isomorphism, where by definition F transforms formal disjoint unions into products. One can get rid of these imprecise formal disjoint unions using the Yoneda embedding $h\colon \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ which freely adds colimits to C. For U a covering family as above, $C_*(U)$ is really an object in $\mathsf{s}(\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \downarrow h_x)$, and the presheaf F is a sheaf if and only if the presheaf on $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ that it represents identifies h_x with the colimit of $C_*(U)$; explicitly, this means that the map

$$F(x) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(h_x, F) \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\varinjlim C_*(U), F) = \varprojlim \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(C_*(U), F)$$

is an isomorphism. (Since $\varinjlim = \pi_0$ and $\varprojlim = \pi^0$, limits and colimits here are really equalizers and coequalizers.)

To understand hypercovers it is useful to consider first a covering family $\{u \to x\}$ consisting of a single morphism, so that there is no need to embed C in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ to express the descent condition relative to $u \to x$. The simplicial object C_* in $(\mathsf{C} \downarrow x)$ is

$$\cdots \substack{\to\\\to\\\to\\\to} u \times_x u \times_x u \substack{\to\\\to\\\to} u \times_x u \rightrightarrows u \to x$$

where only the faces are displayed. The crucial observation is that this simplicial object is determined inductively up to isomorphism by the condition that the canonical maps

$$C_n \to \mathrm{cosk}_{n-1}(C_*)_n$$

be isomorphisms in $(\mathsf{C}\downarrow x)$ for all $n \geq 1$ (and it is by hypothesis a covering map for $n = 0$). Here $\mathrm{cosk}_n\colon \mathsf{s}(\mathsf{C}\downarrow x) \to \mathsf{s}(\mathsf{C}\downarrow x)$ is the nth coskeleton functor. A *representable hypercover* of x in the site (C, τ) is defined to be a simplicial object C_* in the site $(\mathsf{C}\downarrow x)$ (with the induced topology) such that for every $n \geq 0$ the canonical map

$$C_n \to \mathrm{cosk}_{n-1}(C_*)_n$$

is (not necessarily an isomorphism but) a covering map, i.e., generates a covering sieve.

The general definition is the following. First we endow $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ with the topology for which a family of presheaves $\{F_i \to G\}_i$ is a covering if the map

$$\coprod_i F_i \to G$$

induces an epimorphism between the associated sheaves, or equivalently has the local surjectivity property. This topology extends the one on C since a family $\{x_i \to x\}_i$ is a covering family if and only if $\{h_{x_i} \to h_x\}_i$ is a covering family: in this case the local surjectivity property says that the pullback of the sieve generated by $\{x_i \to x\}_i$ along every $y \to x$ is a covering sieve, and by the axioms for a topology this is the case if and only if $\{x_i \to x\}_i$ is a covering family. A *hypercover* of x in C is a representable hypercover C_* of h_x in the site $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$, such that each C_n is a small coproduct of representables. Clearly Čech nerves are hypercovers. With this definition of hypercover, it is still true that a presheaf is a sheaf if and only if it satisfies descent with respect to all hypercovers. Indeed, suppose that F is a sheaf, that $C_* \to h_x$ is a hypercover, and that $\check{C}_* \to h_x$ is the Čech nerve generated by the covering map $C_0 \to h_x$; then one has a commutative diagram

$$\begin{array}{ccccc} C_1 & \rightrightarrows & C_0 & \longrightarrow & h_x \\ \downarrow & & \| & & \| \\ \check{C}_1 & \rightrightarrows & \check{C}_0 & \longrightarrow & h_x \end{array}$$

in which the leftmost vertical arrow is a covering map, and since F is a sheaf $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(?, F)$ transforms covering maps into monomorphisms. Although generalising from Čech covers to hypercovers is not necessary for sheaves of sets, the correct characterization of *simplicial* sheaves must require descent with respect to all hypercovers, not just Čech nerves (see [**DHI04**]; the only differences is that one uses the simplicial Yoneda embedding into $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ and a family of maps in this category is defined to be a covering family if one obtains a covering family in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ by taking connected components, see below). What makes this the "correct characterization" will become clear in the next section.

It is now straightforward to formulate the correct definition of hypercovers in a model site. Let (C, τ) be a model site. We first define the functor of *connected components* $\pi_0^\tau \colon \operatorname{Ho} \mathsf{C}^\wedge \to \operatorname{Sh}(\operatorname{Ho} \mathsf{C})$ from prestacks to sheaves of sets on $\operatorname{Ho} \mathsf{C}$ (recall that by definition τ is a topology on this homotopy category). For $F \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}$, define $\pi_0^\tau(F)$ to be the sheaf associated to the presheaf $x \mapsto \pi_0((RF)(x))$ where R is a fibrant replacement functor on $\mathsf{C}^\wedge$. This is a well-defined presheaf on $\operatorname{Ho} \mathsf{C}$ since RF, being fibrant in $\mathsf{C}^\wedge$, preserves equivalences. Moreover, this presheaf does not depend on the fibrant replacement functor R since different fibrant replacements of F are equivalent in $\mathsf{C}^\wedge$ and hence pointwise equivalent by [**Hir03**, 3.3.5 (1)]. Thus we obtain a well-defined functor $\mathsf{C}^\wedge \to \operatorname{Sh}(\operatorname{Ho} \mathsf{C})$. Now if $F \to G$ is an equivalence in $\mathsf{C}^\wedge$, $RF \to RG$ is a pointwise equivalence by [**Hir03**, 3.3.5 (1)] and in particular $\pi_0(RF(?)) \to \pi_0(RG(?))$ is an isomorphism of presheaves. By the universal property of the homotopy category we get a functor $\pi_0^\tau \colon \operatorname{Ho} \mathsf{C}^\wedge \to \operatorname{Sh}(\operatorname{Ho} \mathsf{C})$. Define a map $F \to G$ in $\operatorname{Ho} \mathsf{C}^\wedge$ to be a *τ-covering map* if $\pi_0^\tau(F) \to \pi_0^\tau(G)$ is an epimorphism of sheaves of sets. A map in $\mathsf{C}^\wedge$ will also be called a τ-covering map if its image in $\operatorname{Ho} \mathsf{C}^\wedge$ is.

Lemma 13. *Let (C, τ) be a model site. A family of morphisms $\{y_i \to x\}_i$ in $\operatorname{Ho} \mathsf{C}$ is a τ-covering family if and only if $\coprod_i^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \to \mathbf{R}\underline{h}_x$ is a τ-covering map.*

Proof. Let $h' \colon \operatorname{Ho} \mathsf{C} \to \operatorname{Sh}(\operatorname{Ho} \mathsf{C})$ be the Yoneda embedding $\operatorname{Ho} \mathsf{C} \to \mathsf{Set}^{(\operatorname{Ho} \mathsf{C})^{\mathrm{op}}}$ composed with the associated sheaf functor. Up to a natural isomorphism, h' is the composite

$$\operatorname{Ho} \mathsf{C} \xrightarrow{\mathbf{R}\underline{h}} \operatorname{Ho} \mathsf{C}^\wedge \xrightarrow{\pi_0^\tau} \operatorname{Sh}(\operatorname{Ho} \mathsf{C}).$$

We already know that $\{y_i \to x\}_i$ is a covering family if and only if $\coprod_i h'_{y_i} \to h'_x$ is an epimorphism. We complete the proof by showing that $\pi_0^\tau \colon \operatorname{Ho} \mathsf{C}^\wedge \to \operatorname{Sh}(\operatorname{Ho} \mathsf{C})$ commutes with coproducts (recall that coproducts indexed by a set I in $\operatorname{Ho} \mathsf{C}$ are derived coproducts under the isomorphism $\mathrm{Ho}(\mathsf{C}^I) \cong (\mathrm{Ho}\,\mathsf{C})^I$). Factor π_0^τ as $\pi_0'\colon \mathrm{Ho}\,\mathsf{C}^\wedge \to \mathsf{Set}^{(\mathrm{Ho}\,\mathsf{C})^{\mathrm{op}}}$ followed by the associated sheaf functor. The latter preserves colimits (being left adjoint), so it suffices to prove that π_0' preserves coproducts. Let $(F_i)_{i\in I}$ be an arbitrary family of objects in $\mathrm{Ho}\,\mathsf{C}^\wedge$. We may suppose that each F_i is fibrant and cofibrant. Then their coproduct in $\mathrm{Ho}\,\mathsf{C}^\wedge$ coincides with their coproduct $G = \coprod_i F_i$ in $\mathsf{C}^\wedge$. Since each F_i is a prestack and coproducts of simplicial sets preserve equivalences, G is a prestack. Therefore, a fibrant replacement $G \to RG$ is just a pointwise fibrant replacement. Using that $\pi_0\colon \mathsf{sSet} \to \mathsf{Set}$ preserves colimits (it is left adjoint to the inclusion), we find $\pi_0'(G)(z) = \pi_0((RG)(z)) = \pi_0(G(z)) = \coprod_i \pi_0(F_i(z)) = \coprod_i \pi_0'(F_i)(z)$. □

We shall consider for $G \in \mathsf{C}^\wedge$ the comma category $(\mathsf{C}^\wedge \downarrow G)$ which we endow with the induced model structure (equivalences, fibrations, and cofibrations are as in $\mathsf{C}^\wedge$). The forgetful functor induces a functor $\mathrm{Ho}(\mathsf{C}^\wedge \downarrow G) \to \mathrm{Ho}\,\mathsf{C}^\wedge$. Fix a fibrant object $x \in \mathsf{C}$. Since $\mathsf{C}^\wedge$ and hence $(\mathsf{C}^\wedge \downarrow \underline{h}_x)$ have all limits and colimits, one can certainly define the skeleton and coskeleton functors $\mathrm{sk}_n\colon \mathsf{s}(\mathsf{C}^\wedge \downarrow \underline{h}_x) \to \mathsf{s}(\mathsf{C}^\wedge \downarrow \underline{h}_x)$ and $\mathrm{cosk}_n\colon \mathsf{s}(\mathsf{C}^\wedge \downarrow \underline{h}_x) \to \mathsf{s}(\mathsf{C}^\wedge \downarrow \underline{h}_x)$. Recall from Proposition 10 that this is a Quillen adjunction for the Reedy model structure. A *τ-hypercover* of x in C is a simplicial object C_* in $(\mathsf{C}^\wedge \downarrow \underline{h}_x)$ such that

- for every $n \geq 0$, the image in $\mathrm{Ho}\,\mathsf{C}^\wedge$ of the canonical map $C_n \to \mathbf{R}\,\mathrm{cosk}_{n-1}(C_*)_n$ is a τ-covering map and
- each C_n is equivalent in $\mathsf{C}^\wedge$ to a small coproduct of representables.

If x is not fibrant, a τ-hypercover of x in C is defined to be a hypercover of some fibrant replacement of x; it is therefore an object of $\mathsf{s}(\mathsf{C}^\wedge \downarrow \mathbf{R}\underline{h}_x)$. Observe that in $\mathrm{Ho}\,\mathsf{C}^\wedge$ a coproduct of representables is the same thing as a homotopy coproduct of $\mathbf{R}\underline{h}_y$'s, because h_y is a cofibrant replacement of $\mathbf{R}\underline{h}_y$ (Proposition 12). Thus a hypercover of x is an augmented simplicial object of the form

$$\cdots \substack{\to\\\to\\\to\\\to} \coprod_{i\in I_2}^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \substack{\to\\\to\\\to} \coprod_{i\in I_1}^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \rightrightarrows \coprod_{i\in I_0}^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \to \mathbf{R}\underline{h}_x. \tag{12}$$

The reason that we need x to be fibrant in this definition is the following. One could define a model topology $\tau^\wedge$ on $\mathsf{C}^\wedge$ such that the τ-covering maps defined above are exactly the maps generating a $\tau^\wedge$-covering sieve. If $\underline{h}_x$ is not fibrant one cannot necessarily pull back $\tau^\wedge$ through the functor $\mathrm{Ho}(\mathsf{C}^\wedge \downarrow \underline{h}_x) \to \mathrm{Ho}\,\mathsf{C}^\wedge$ (see Lemma 14), which is what we really do in the first condition above.

We discuss two especially useful kinds of hypercovers. A *representable hypercover* is a hypercover of the form $\mathbf{R}\underline{h}_{y_*} \to \mathbf{R}\underline{h}_x$ induced by an augmented simplicial object $y_* \to x$ in C. In this case we also say that $y_* \to x$ a τ-hypercover. Using Lemma 13 and the fact that $\underline{h}$ commutes with limits and hence with coskeletons, we obtain the following characterization of representable hypercovers. An augmented simplicial object $y_* \to x$ is a τ-hypercover if and only if for every $n \geq 0$ the canonical map $y_n \to \mathbf{R}\,\mathrm{cosk}_{n-1}(y_*)_n$ in $\mathrm{Ho}\,\mathsf{C}$ generates a τ-covering sieve.

If $U = \{y_i \to x\}_i$ is a τ-covering family, then by Lemma 13 $\coprod_i^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \to \mathbf{R}\underline{h}_x$ is a τ-covering map and we define inductively a τ-hypercover C_*, called the *Čech hypercover* associated to U or the *homotopy nerve* of the covering U, by

$$C_0 = \coprod_i^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i} \quad \text{and} \quad C_n = \mathbf{R}\,\mathrm{cosk}_{n-1}(C_*)_n.$$

(Here $\mathbf{R}\,\mathrm{cosk}_{n-1}(C_*)$ really means $\mathbf{R}(i_{n-1})_*(i_{n-1}^* C_*)$ where $\mathsf{s}_{n-1}(\mathsf{C} \downarrow \mathbf{R}\underline{h}_x)$ is given the Reedy model structure. The construction is well-defined up to a pointwise equivalence.) Since $\underline{h}$ commutes with limits, $C_* \to \mathbf{R}\underline{h}_x$ has the form

$$\cdots \substack{\to\\\to\\\to\\\to} \coprod_{i_0,i_1,i_2}^{\mathbf{L}} \mathbf{R}\underline{h}(y_{i_0} \times_x^{\mathbf{R}} y_{i_1} \times_x^{\mathbf{R}} y_{i_2}) \substack{\to\\\to\\\to} \coprod_{i_0,i_1}^{\mathbf{L}} \mathbf{R}\underline{h}(y_{i_0} \times_x^{\mathbf{R}} y_{i_1}) \rightrightarrows \coprod_{i_0}^{\mathbf{L}} \mathbf{R}\underline{h}(y_{i_0}) \to \mathbf{R}\underline{h}_x.$$

It may seem that since we restricted the values of a hypercover to certain coproducts we should also restrict the face maps and degeneracy maps to be "morphisms of coproducts" (as is the case in a Čech hypercover). This is in fact automatically the case. Precisely, any morphism $h_x \to \coprod_i h_{y_i}$ factors through h_{y_i} for a uniquely determined index i (it is the index of the component into which id_x goes). This follows from the simplicial Yoneda lemma: there are bijections

$$\coprod_i \mathsf{C}^\wedge(h_x, h_{y_i}) \cong \coprod_i h_{y_i}(x)_0 \cong \mathsf{C}^\wedge(h_x, \coprod_i h_{y_i})$$

sending a morphism $h_x \to h_{y_i}$ to the composition $h_x \to h_{y_i} \to \coprod_i h_{y_i}$. Thus an arbitrary morphism $\coprod_i h_{x_i} \to \coprod_j h_{y_j}$ is induced by an element of

$$\prod_i \coprod_j \mathsf{C}^\wedge(h_{x_i}, h_{y_j}) = \prod_i \coprod_j \mathsf{C}(x_i, y_j).$$

2.4 Stacks

In the classical situation of presheaves of sets on a category C, a topology τ on C allows us to define a sheaf as a presheaf F which satisfies the following descent condition: for any covering family $U = \{x_i \to x\}_i$ of an object x, the map

$$F(x) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(h_x, F) \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\varinjlim C_*(U), F) = \varprojlim \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(C_*(U), F) \tag{13}$$

is an isomorphism, where $C_*(U)$ is the Čech cover associated to the covering family $\{x_i \to x\}_i$ (as we already mentioned, this will then hold for arbitrary hypercovers). The category $\mathrm{Sh}(\mathsf{C})$ is the full subcategory of $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ consisting of sheaves. A basic result is that the inclusion $i\colon \mathrm{Sh}(\mathsf{C}) \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ has a left adjoint left inverse a, called the associated sheaf functor. Moreover, the counit $ia \to \mathrm{id}$ of this adjunction is a τ-local isomorphism, where a map of presheaves $F \to G$ is called a τ-local isomorphism if for any $x \in \mathsf{C}$ there exists a τ-covering sieve S such that $F(u) \to G(u)$ is an isomorphism for all $u \to x$ in S. These formal properties imply at once that the category of sheaves, together with the functor a, is a localization of the category of presheaves along τ-local isomorphisms. Indeed, if $f\colon \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{D}$ is a functor that sends τ-local isomorphisms to isomorphisms, then $fia \cong f$, and any functor $g\colon \mathrm{Sh}(\mathsf{C}) \to \mathsf{D}$ satisfying $ga = f$ must be fi because $g = gai = fi$. Writing presheaves as colimits of representables, it is not difficult to prove that the functor a is also universal among colimit-preserving functors on $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ that send $\varinjlim C_* \to h_x$ to an isomorphism for every x and every hypercover $C_* \to h_x$.

These three descriptions of the category of sheaves of sets on a site (the descent property, the localization with respect to τ-local isomorphisms, and the cocontinuous localization with respect to hypercovers) have analoguous counterparts in the context of prestacks. We shall follow the second one to define the model category of *stacks*: it will be the localization of $\mathsf{C}^\wedge$ along τ-local equivalences, which are to equivalences as local isomorphisms were to isomorphisms. Here of course localization must be understood in the context of model categories, as Bousfield localization. There are several ways to define τ-local equivalences. Our official definition can be summarized as follows: a morphism is a τ-local equivalence if it induces τ-local isomorphisms on all presheaves of homotopy groups. To make this precise we need a lemma.

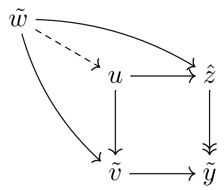
Lemma 14. *Let (C, τ) be a model site and let $x \in \mathsf{C}$ be fibrant. There is a model topology on $(\mathsf{C}{\downarrow}x)$ for which a sieve is a covering sieve if and only if its image by the functor $\mathrm{Ho}(\mathsf{C}{\downarrow}x) \to \mathrm{Ho}\,\mathsf{C}$ generates a τ-covering sieve.*

Proof. We first note that for an arbitrary functor $\phi\colon \mathsf{D} \to \mathsf{E}$ where E is a site, the sieves in D whose images by ϕ generate covering sieves always satisfy all the axioms for a Grothendieck topology except possibly the stability axiom. This axiom reads: for any $f\colon z \to y$ in D and any covering sieve S of y, $f^*(S) = \{g \mid fg \in S\}$ is a covering sieve of z. Let us prove that this axiom holds when $\mathsf{D} = \mathrm{Ho}(\mathsf{C}\downarrow x)$, $\mathsf{E} = \mathrm{Ho}\,\mathsf{C}$, and ϕ is induced by the forgetful functor $(\mathsf{C}\downarrow x) \to \mathsf{C}$. Let S be a sieve on $y \to x$ and let $f\colon (z \to x) \to (y \to x)$ be a morphism in $\mathrm{Ho}(\mathsf{C}\downarrow x)$. The hypothesis is that the sieve generated by $\phi(S)$ is a covering sieve of y, and one must prove that the sieve generated by $\phi(f^*(S))$ is a covering sieve of z. It will suffice to prove that

$$\phi(f)^*(\text{sieve generated by } \phi(S)) \subset \text{sieve generated by } \phi(f^*(S))$$

since the left-hand side is a covering sieve and the other inclusion is obvious. An arbitrary morphism $g\colon w \to z$ on the left is such that $\phi(f)g = \phi(h)k$ for some $h \in S$ and $k\colon w \to v$. We must find morphisms m in $\mathrm{Ho}(\mathsf{C}\downarrow x)$ and n in $\mathrm{Ho}\,\mathsf{C}$ such that $g = \phi(m)n$ and $fm \in S$.

Let us abbreviate an object $y \to x$ in $(\mathsf{C}\downarrow x)$ to y_x. We choose once and for all an isomorphism $y_x \cong \tilde{y}_x$ in $\mathrm{Ho}(\mathsf{C}\downarrow x)$ where $\tilde{y}_x$ is both fibrant and cofibrant. The forgetful functor $(\mathsf{C}\downarrow x) \to \mathsf{C}$ obviously preserves cofibrant objects, and since x is fibrant it also preserves fibrant objects. Thus $\tilde{y}$ is also fibrant and cofibrant. Define similarly $\tilde{z}_x$, $\tilde{w}_x$, and $\tilde{v}_x$. The induced maps $\tilde{f}\colon \tilde{z}_x \to \tilde{y}_x$ and $\tilde{h}\colon \tilde{v}_x \to \tilde{y}_x$ are represented by maps in $(\mathsf{C}\downarrow x)$; factor the first one into a trivial cofibration $\tilde{z}_x \to \hat{z}_x$ followed by a fibration $\hat{z}_x \to \tilde{y}_x$, and denote by u_x the pullback $\hat{z}_x \times_{\tilde{y}_x} \tilde{v}_x$ in $(\mathsf{C}\downarrow x)$. Observe that $\hat{z}_x$ is fibrant and cofibrant. The maps $\tilde{w} \to \hat{z}$ and $\tilde{w} \to \tilde{v}$ in $\mathrm{Ho}\,\mathsf{C}$ are represented by maps in C, and we have a diagram in C



in which the square is a pullback (the forgetful functor $(\mathsf{C}\downarrow x) \to \mathsf{C}$ is a right adjoint and hence preserves pullbacks). The two maps from $\tilde{w}$ to $\tilde{y}$ become equal in the homotopy category and so they are homotopic. By [**Hir03**, 7.3.12 (2)], one can replace $\tilde{w} \to \hat{z}$ by a homotopic map (i.e., another representative of the same map in $\mathrm{Ho}\,\mathsf{C}$) that makes the boundary of the above diagram strictly commutative, and we get a map $\tilde{w} \to u$ in C as shown above. If m is the composite $u_x \to \hat{z}_x \cong z_x$ in $\mathrm{Ho}(\mathsf{C}\downarrow x)$ and n is the composite $w \cong \tilde{w} \to u$ in $\mathrm{Ho}\,\mathsf{C}$, all this implies that $g = \phi(m)n$. It remains to prove that $fm \in S$. But fm is the composition of $u_x \to \tilde{v}_x \cong v_x$ and h, so we are done. □

Let (C, τ) be a model site and let x be a fibrant object in C. We continue to write τ for the model topology of the lemma on $(\mathsf{C}\downarrow x)$. Let $s\colon h_x \to F$ be a morphism in $\mathrm{Ho}\,\mathsf{C}^\wedge$ (or equivalently, by the derived Yoneda lemma, a connected component of $(RF)(x)$ for some fibrant replacement RF of F). For $n \geq 1$, we define the nth *homotopy sheaf* of F pointed at s to be the sheaf on $\mathrm{Ho}(\mathsf{C}\downarrow x)$ associated to the presheaf

$$(u\colon y \to x) \mapsto \pi_n((RF)(y), sh_u).$$

Here $sh_u\colon h_y \to F$ is a morphism in $\mathrm{Ho}\,\mathsf{C}^\wedge$ that one identifies with a connected component of $(RF)(y)$. This presheaf is well-defined on $(\mathsf{C}\downarrow x)$ and descends to $\mathrm{Ho}(\mathsf{C}\downarrow x)$ for exactly the same reasons as the presheaf of connected component defined in §2.3. We denote the resulting sheaf by $\pi_n^\tau(F, s)$. This defines for each fibrant x and each $n \geq 1$ a functor

$$\pi_n^\tau\colon (h_x \downarrow \mathrm{Ho}(\mathsf{C}^\wedge)) \to \mathrm{Sh}(\mathrm{Ho}(\mathsf{C}\downarrow x))$$

which obviously factors through the category of sheaves of groups on $\mathrm{Ho}(\mathsf{C}\downarrow x)$ and even of abelian groups if $n \geq 2$.

A morphism $f\colon F \to G$ in $\mathsf{C}^\wedge$ is called a *τ-local equivalence* if

- $\pi_0^\tau(f)$ is an isomorphism;
- for any fibrant $x \in \mathsf{C}$ and any morphism $s\colon h_x \to F$ in $\mathrm{Ho}\,\mathsf{C}^\wedge$ the map $\pi_n^\tau(f)\colon \pi_n^\tau(F, s) \to \pi_n^\tau(G, fs)$ is an isomorphism.

Because of the first condition any τ-local equivalence is in particular a τ-covering map.

Just as it is the case for simplicial sets, it is possible to give a more compact "basepoint-free" definition of local equivalences: a morphism $f\colon F \to G$ is a τ-local equivalence if and only if for each $n \geq 0$ the induced map

$$F \to \mathbf{R}\,\mathrm{cosk}_{n-1}(f)_n$$

is a τ-covering map in $\mathrm{Ho}\,\mathsf{C}^\wedge$, where f is viewed as a constant simplicial object in $\mathsf{s}(\mathsf{C}^\wedge\downarrow G)$. We do not give a proof of this fact but it is essentially the same as the proof of the corresponding fact for classical simplicial presheaves (see [**Jar87**, Thm. 1.12]): one must first interpret this condition as the right τ-local lifting property of a fibration replacement of f with respect to the inclusions $\partial\Delta^n \subset \Delta^n$, and the latter is seen to be equivalent to f being a weak equivalence (when C is a point this is just the fact that $\partial\Delta^n \subset \Delta^n$ are generating cofibrations in sSet).

Let (C,τ) be a model site. The *model category of stacks* on (C,τ) is the left Bousfield localization of $\mathsf{C}^\wedge$ along τ-local equivalences; it is denoted by $\mathsf{C}^{\sim,\tau}$. The existence of this left Bousfield localization is not obvious because τ-local equivalences do not form a sufficiently small set for the general existence theorem to apply. The definition of $\mathsf{C}^{\sim,\tau}$ in [**HAGI**] uses a different route to overcome this problem, and only afterwards is it proved that it is a left Bousfield localization along τ-local equivalences. Here we shall assume that this localization exists. Again, $\mathsf{C}^{\sim,\tau}$ is a left proper, cellular, and simplicial model category. By the theory of Bousfield localizations, the cofibrations (resp. the trivial fibrations) in $\mathsf{C}^{\sim,\tau}$ are the projective cofibrations (resp. the projective trivial fibrations). The identity $\mathrm{id}\colon \mathsf{C}^\wedge \to \mathsf{C}^{\sim,\tau}$ is a left Quillen functor, and its derived right adjoint $\mathbf{R}\mathrm{id}\colon \operatorname{Ho}\mathsf{C}^{\sim,\tau} \to \operatorname{Ho}\mathsf{C}^\wedge$ is fully faithful; its essential image consists of those objects that are equivalent in $\mathsf{C}^\wedge$ to fibrant objects in $\mathsf{C}^{\sim,\tau}$. The functor $\mathbf{L}\mathrm{id}\colon \mathsf{C}^\wedge \to \mathsf{C}^{\sim,\tau}$ is called the *associated stack functor*; it is left inverse to $\mathbf{R}\mathrm{id}$. One often identifies $\operatorname{Ho}\mathsf{C}^{\sim,\tau}$ with a subcategory of $\operatorname{Ho}\mathsf{C}^\wedge$, and with this identification the associated stack functor is just induced by a fibrant replacement functor in $\mathsf{C}^{\sim,\tau}$.

It turns out that stacks can be characterized among prestacks in exactly the same way as sheaves are characterized among presheaves (if one is willing to use all hypercovers and not just Čech covers). We say that a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}$ *has hyperdescent* if for every hypercover C_* of x the map

$$(RF)(x) \cong \mathbf{R}\operatorname{Map}(\mathbf{R}\underline{h}_x, F) \to \mathbf{R}\operatorname{Map}(\underrightarrow{\operatorname{holim}}\, C_*, F) \cong \underleftarrow{\operatorname{holim}}\, \mathbf{R}\operatorname{Map}(C_*, F),$$

induced by the map $\underrightarrow{\operatorname{holim}}\, C_* \to \mathbf{R}\underline{h}_x$ in $\operatorname{Ho}\mathsf{C}^\wedge$ adjoint to $C_* \to \mathbf{R}\underline{h}_x$ in $\operatorname{Ho}\mathsf{sC}^\wedge$, is an isomorphism in $\operatorname{Ho}\mathsf{sSet}$. If $C_* \to \mathbf{R}\underline{h}_x$ is of the form (12) and if $F \in \mathsf{C}^\wedge$ if fibrant, then by the derived Yoneda lemma $\mathbf{R}\operatorname{Map}(C_*, F)$ is the cosimplicial simplicial set

$$\prod_{i\in I_0} F(y_i) \rightrightarrows \prod_{i\in I_1} F(y_i) \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \prod_{i\in I_2} F(y_i) \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \cdots.$$

We omit the rather complicated proof of the next theorem which can be found in [**HAGI**].

Theorem 15. *Let (C,τ) be a model site. Then $\mathsf{C}^{\sim,\tau}$ is the left Bousfield localization of $\mathsf{C}^\wedge$ with respect to the set of morphisms $\underrightarrow{\operatorname{holim}}\, C_* \to \mathbf{R}\underline{h}_x$ where $C_* \to \mathbf{R}\underline{h}_x$ runs through τ-hypercovers. Equivalences in $\mathsf{C}^{\sim,\tau}$ are exactly the τ-local equivalences, and fibrant objects are exactly the fibrant objects in $\mathsf{C}^\wedge$ having hyperdescent.*

In the proof one actually defines $\mathsf{C}^{\sim,\tau}$ as the left Bousfield localization of $\mathsf{C}^\wedge$ along the maps $\underrightarrow{\operatorname{holim}}\, C_* \to \mathbf{R}\underline{h}_x$ associated to sufficiently few hypercovers $C_* \to \mathbf{R}\underline{h}_x$, and one proves that the equivalences in $\mathsf{C}^{\sim,\tau}$ are exactly the τ-local equivalences. The basic step of the proof is the observation that a morphism $\underline{h}_x \to \underline{h}_y$ (x and y fibrant) is a τ-local equivalence if and only if it is a hypercover when viewed as an object in $\mathsf{s}(\mathsf{C}^\wedge \downarrow \underline{h}_y)$, which follows at once from the second description of τ-local equivalences. The last part of the theorem is just the characterization of fibrant objects in left Bousfield localizations of left proper model categories that we already used. This characterization also implies that a fibrant object $F \in \mathsf{C}^\wedge$ has hyperdescent if and only if for every τ-local equivalence $G \to H$ the induced map

$$\mathbf{R}\operatorname{Map}(H, F) \to \mathbf{R}\operatorname{Map}(G, F) \tag{14}$$

is an isomorphism in $\operatorname{Ho}\mathsf{sSet}$. Conversely, a map $G \to H$ is a τ-local equivalence if and only if, for every object $F \in \mathsf{C}^\wedge$ having hyperdescent, (14) is an isomorphism (this is a general characterization of equivalences in a model category which is also a direct consequence of the derived Yoneda lemma).

Call a hypercover $C_* \to \mathbf{R}\underline{h}_x$ *finite* if each C_n is a finite coproduct of representables. Recall that a topology is *quasi-compact* if every covering sieve contains a covering sieve generated by a finite family.

Proposition 16. *Let (C, τ) be a model site such that τ is quasi-compact. Then $\mathsf{C}^{\sim,\tau}$ is the left Bousfield localization of $\mathsf{C}^\wedge$ with respect to the set of morphisms $\underrightarrow{\operatorname{holim}}\, C_* \to \mathbf{R}\underline{h}_x$ where $C_* \to \mathbf{R}\underline{h}_x$ runs through finite τ-hypercovers.*

In view of Theorem 15 and our previous results on prestacks, $\operatorname{Ho}\mathsf{C}^{\sim,\tau}$ is equivalent to the full subcategory of $\operatorname{Ho}\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ consisting of the functors $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}$ such that

- F preserves weak equivalences and
- F has hyperdescent.

When these conditions are satisfied, we say that F is a *stack*.

To avoid confusing the two model structures $\mathsf{C}^\wedge$ and $\mathsf{C}^{\sim,\tau}$ we use henceforth the following rules. We continue to write $\mathbf{R}\operatorname{Map}(F, G)$ for mapping spaces in $\mathsf{C}^\wedge$ and we use $\mathbf{R}_\tau \operatorname{Map}(F, G)$ for the mapping spaces in $\mathsf{C}^{\sim,\tau}$. Similarly, we shall use the letter R for fibrant replacements in $\mathsf{C}^\wedge$ while R_τ will be used for fibrant replacements in $\mathsf{C}^{\sim,\tau}$ (we make no such distinction for cofibrant replacements since a cofibrant replacement in $\mathsf{C}^\wedge$ is in particular a cofibrant replacement in $\mathsf{C}^{\sim,\tau}$). Thus

$$\mathbf{R}\operatorname{Map}(F, G) = \operatorname{Map}(QF, RG) \quad \text{and} \quad \mathbf{R}_\tau \operatorname{Map}(F, G) = \operatorname{Map}(QF, R_\tau G),$$

the canonical map $\mathbf{R}\operatorname{Map}(F, G) \to \mathbf{R}_\tau \operatorname{Map}(F, G)$ being an isomorphism for all F if and only if G has hyperdescent.

We remark that the identity $\mathsf{C}^\wedge \to \mathsf{C}^{\sim,\tau}$ preserves homotopy colimits since it is left Quillen. It follows that if $C_* \to \mathbf{R}\underline{h}_x$ is a hypercover and $C'_* \to \mathbf{R}\underline{h}_x$ is any object in $\mathsf{s}(\mathsf{C}^\wedge \downarrow \mathbf{R}\underline{h}_x)$ that is levelwise τ-locally equivalent to $C_* \to \mathbf{R}\underline{h}_x$, then $\underrightarrow{\operatorname{holim}}\, C'_* \to \mathbf{R}\underline{h}_x$ is an isomorphism in $\operatorname{Ho}\mathsf{C}^{\sim,\tau}$.

We say that a model topology τ on C is *subcanonical* if, for any $x \in \mathsf{C}$, the prestack $\mathbf{R}\underline{h}_x$ is a stack (i.e., has hyperdescent). This means that the derived Yoneda embedding factors through $\operatorname{Ho}\mathsf{C}^{\sim,\tau}$ as in

$$\begin{array}{ccc} \operatorname{Ho}\mathsf{C} & \xrightarrow{\mathbf{R}\underline{h}} & \operatorname{Ho}\mathsf{C}^\wedge \\ & \searrow & \uparrow{\scriptstyle \mathbf{R}\mathrm{id}} \\ & & \operatorname{Ho}\mathsf{C}^{\sim,\tau}. \end{array}$$

The category $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, viewed as a monoidal category for the direct product, is closed. This is a general fact about presheaves of enriched categories, and the exponential $\operatorname{Hom}(F, G)$ of two such presheaves is given by

$$\operatorname{Hom}(F, G)(x) = \operatorname{Map}(F \times h_x, G)$$

where h is the enriched Yoneda embedding. As explained in [**HAGI**, 3.6], $\mathsf{C}^{\sim,\tau}$ need not be a monoidal model category. However, there exists another model structure on $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$, called the *injective model structure*, whose equivalences are also the τ-local equivalences and which is compatible with this monoidal structure. This model structure is simply the left Bousfield localization of $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ along the same set of morphisms that was used to define $\mathsf{C}^{\sim,\tau}$, but now we endow $\mathsf{sSet}^{\mathsf{C}^{\mathrm{op}}}$ with the model structure in which equivalences and cofibrations are defined objectwise. It follows that the homotopy category $\operatorname{Ho}\mathsf{C}^{\sim,\tau}$ is cartesian closed, and its exponentials can be computed by

$$\mathbf{R}\operatorname{Hom}(F, G) = \operatorname{Hom}(F, R_{\mathrm{inj}} G)$$

where $R_{\mathrm{inj}} G$ is a fibrant replacement of G for the injective model structure (and F is a cofibrant replacement of itself since left Bousfield localization does not alter cofibrations).

3 Derived algebraic geometry

3.1 Introduction

Let us first recall the definition of a scheme over a base commutative ring k, from the functorial point of view. The category Aff_k of affine k-schemes is defined to be the opposite of the category Comm_k of commutative k-algebras (associative and with unit). We write suggestively $X = \operatorname{Spec} A$ to mean that X is the object of Aff_k corresponding to the algebra A. Let h denote the Yoneda embedding of Aff_k into the category of presheaves of sets on Aff_k. A morphism $f\colon A \to B$ between k-algebras is called a *Zariski open immersion* if it is flat, if $f^*\colon \mathsf{Mod}_B \to \mathsf{Mod}_A$ is fully faithful, and if the functor $(A \downarrow \mathsf{Comm}_k)(B, ?)$ preserves filtered colimits. The category Aff_k is endowed with a Grothendieck topology, called the Zariski topology, generated by the finite families $\{Y_i \to X\}_i$ of Zariski open immersions such that the preimages of the prime spectra (i.e., the sets of prime ideals) of the Y_i cover the prime spectrum of X. The Zariski topology is subcanonical, that is, representable presheaves have effective descent relative to Zariski coverings, so we have a fully faithful embedding h from affine k-schemes to sheaves on the site of affine k-schemes. Now if $Y \to h(\operatorname{Spec} A)$ is any monomorphism of sheaves, call it a Zariski open immersion if there exists Zariski open immersions $A \to B_i$ such that Y, viewed as a subfunctor of $h(\operatorname{Spec} A)$, is the image of $\coprod_i h(\operatorname{Spec} B_i) \to h(\operatorname{Spec} A)$. Finally, a general morphism $Y \to X$ between sheaves is a Zariski open immersion if it becomes so after pulling back along any morphism $h(\operatorname{Spec} A) \to X$. The category Sch of schemes is then the full subcategory of sheaves of sets on the Zariski site Aff_k whose objects X are locally affine in the following sense: there exists affine schemes Y_i and Zariski open immersions $h(Y_i) \to X$ such that the induced map $\coprod_i h(Y_i) \to X$ is an epimorphism.

The Zariski topology has its origin in the geometric point of view for schemes, where it is actually the name of a classical topology on the prime spectrum $\operatorname{Spec} A$ of a k-algebra A. In this topology an open set $D(I)$ is the set of prime ideals which do not contain a given subset I of A. The topological space $\operatorname{Spec} A$ has a canonical sheaf of k-algebras whose stalks are local rings, namely the one associated to the presheaf $D(I) \mapsto S(I)^{-1}A$, where $S(I)$ is the set of elements of A which do not belong to any element of $D(I)$. With this point of view a geometric scheme is a locally k-ringed space covered by open affine schemes. The full subcategory of geometric schemes that are isomorphic to spectra is equivalent to the category Aff_k of the previous paragraph. Since any geometric schemes is a colimit of spectra by definition, the functor that restricts a presheaf on the category of geometric scheme to a presheaf on the category Aff_k is a fully faithful embedding. Precomposing with the Yoneda embedding, we obtain a fully faithful embedding of the category of geometric schemes into the category of presheaves on Aff_k. Its essential image is exactly the category of schemes, and the Zariski open immersions correspond precisely to the open immersions of ringed spaces.

Put simply, homotopical algebraic geometry has vocation to replace the category of commutative k-algebras in the above construction by the category of monoids on an arbitrary monoidal $(\infty, 1)$-category C. Most notions of classical algebraic geometry can be formulated in such a way that they remain meaningful in this more general context. An example of such a reformulation is the definition of Zariski open immersions given above. Classical algebraic geometry is recovered by taking $\mathsf{C} = \mathsf{Mod}_k$ with the trivial ∞-structure. The category Aff_C of affine schemes is defined as the opposite of the category of commutative monoids in C. Then one assumes given an "∞-topology" on Aff_C, and one defines a scheme to be a stack on Aff_C that is obtained by glueing representable stacks using morphisms playing the rôle of Zariski open immersion. More generally, there are analogues to algebraic stacks as well as their higher-categorical versions. All of them are examples of *geometric stacks*.

Our main reference for homotopical algebraic geometry is [**HAGII**]. In this chapter we shall only be interested in the following special case: C is the model category of simplicial k-modules. The resulting geometry is called *derived algebraic geometry*. With the exception of the second half of the proof of Theorem 21 and the proofs of Lemmas 32 and 33, all proofs in this chapter are from [**HAGII**] unless otherwise stated in the text.

3.2 Derived stacks

We start from the base symmetric monoidal model category sMod_k of simplicial k-modules over a commutative ring k. This is a proper simplicial model category whose equivalences and fibrations are defined through the forgetful functor $\mathrm{Map}(k, ?) \colon \mathsf{sMod}_k \to \mathsf{sSet}$, left adjoint to the free simplicial k-module functor $? \otimes k$ (here k is a constant simplicial k-module). Recall from the Dold–Kan equivalence that the homotopy groups of the underlying simplicial set of a simplicial k-module are base-point invariant and agree with the homology groups of the associated nonnegatively graded complex, and that a map $M \to N$ is a fibration if and only if the induced map $M \to \pi_0(M) \times_{\pi_0(N)} N$ is degreewise surjective. In particular degreewise surjective morphisms and morphisms between constant objects are fibrations. The tensor product is defined levelwise while the internal hom's are given by $\mathrm{Hom}(M, N)_n = \mathsf{sMod}_k(M \otimes_k k[\Delta^n], N)$ with the k-module structure coming from the target.

We let sComm_k be the category of commutative monoids in sMod_k, or in other words the category of simplicial commutative k-algebras. The category sComm_k is a proper simplicial model category whose equivalences and fibrations are defined on the underlying simplicial k-modules (hence on the underlying simplicial sets). If $A \in \mathsf{sComm}_k$, we denote by sMod_A the model category of simplicial A-modules. Equivalences and fibrations are defined on the underlying simplicial k-modules and this is again a proper simplicial model category. The homotopy relation is compatible with the additive structure, so that the localization functor $\mathsf{sMod}_A \to \mathrm{Ho}\,\mathsf{sMod}_A$ is enriched in abelian groups. Moreover, this model category is a monoidal model category for the tensor product $\otimes_A$. This tensor product is *left balanced* in the sense that $M \otimes_A ?$ preserves equivalences as soon as M is cofibrant. If $A \to B$ is a cofibration in sComm_k, we also have that the extension of scalars $? \otimes_A B \colon \mathsf{sMod}_A \to \mathsf{sMod}_B$ preserve equivalences. As a formal consequences of these facts we have the following important result: if $A \to B$ and $A \to C$ are maps in sComm_k then the canonical map in $\mathrm{Ho}\,\mathsf{sMod}_A$ from the underlying A-module of the homotopy pushout of B and C over A to the derived tensor product in sMod_A of B and C is an isomorphism. This is fortunate since the standard notation scheme yields the same notation $B \otimes^{\mathbf{L}}_A C$ for both constructions.

We recall that for K a simplicial set and X an object in any of these simplicial model categories, $K \otimes X$ is the diagonal of the bisimplicial object given in degree (p, q) by

$$\coprod_{x \in K_p} X_q, \tag{15}$$

with horizontal simplicial maps defined from the simplicial structure of K and vertical ones defined from the simplicial structure of X (see [**GJ99**, ch. II, §2]).

A morphism $f \colon A \to B$ in sComm_k gives rise to a Quillen adjunction

$$f_* \colon \mathsf{sMod}_A \rightleftarrows \mathsf{sMod}_B : f^*$$

where f_* is extension of scalars. If f is a weak equivalence, then this adjunction is a Quillen equivalence. Indeed, a map $\phi\colon M \to f^*(N)$ is, as a map of simplicial k-modules, the composition

$$M \cong M \otimes_A A \xrightarrow{M \otimes_A f} M \otimes_A B \xrightarrow{\phi^\flat} N$$

where $M \otimes_A f$ is a weak equivalence if M is cofibrant, in which case the two-out-of-three axiom imply that ϕ is an equivalence if and only if $\phi^\flat$ is.

If A is a commutative simplicial k-algebra, then $\pi_*(A)$ is endowed with a graded k-algebra structure (induced by the shuffle map), and π_* is a functor from simplicial commutative k-algebras to nonnegatively graded k-algebras. In particular, $\pi_0(A)$ is a k-algebra and $\pi_n(A)$ is a $\pi_0(A)$-module for every $n \geq 0$. Similarly, if M is a simplicial A-module the shuffle map endows $\pi_*(M)$ with a structure of graded $\pi_*(A)$-module. Note that the functors π_n preserve finite products and filtered colimits.

Let $A \in \mathsf{sComm}_k$. The functor $\pi_0 \colon \mathsf{sMod}_A \to \mathsf{Mod}_{\pi_0(A)}$ is left adjoint to the functor $i \colon \mathsf{Mod}_{\pi_0(A)} \to \mathsf{sMod}_A$ which associates to a $\pi_0(A)$-module M the constant simplicial $\pi_0(A)$-module $i(M)$, viewed as a simplicial A-module through the canonical projection $A \to \pi_0(A)$.

Moreover, if we endow the category $\mathsf{Mod}_{\pi_0(A)}$ with the trivial model structure, then π_0 preserves all equivalences and cofibrations and so is left Quillen. In particular there is a derived adjunction

$$\mathbf{L}\pi_0\colon \operatorname{Ho}\mathsf{sMod}_A \rightleftarrows \mathsf{Mod}_{\pi_0(A)} :\mathbf{R}i.$$

Since both π_0 and i preserve equivalences, we will often write abusively $\pi_0 = \mathbf{L}\pi_0$ and $i = \mathbf{R}i$. Note that the counit of the underived adjunction is an isomorphism. Since π_0 preserves equivalences the counit of the derived adjunction is an isomorphism as well. Although neither of the functors π_0 and i is a (co)monoidal functor for general A, the right adjoint i has a structure of nonunital monoidal functor: there is a canonical map

$$i(M) \otimes_A i(N) \to i(M \otimes_{\pi_0(A)} N)$$

which is always an isomorphism since $A \to \pi_0(A)$ is surjective in each degree. Adjoint to

$$i(\operatorname{Hom}_{\pi_0(A)}(M,N)) \otimes_A i(M) \to i(\operatorname{Hom}_{\pi_0(A)}(M,N) \otimes_{\pi_0(A)} M) \to i(N)$$

we find a natural map

$$i(\operatorname{Hom}_{\pi_0(A)}(M,N)) \to \operatorname{Hom}_A(i(M), i(N)).$$

We claim that this is also an isomorphism. It is clearly so in degree 0 by definition of the A-module structure on $i(M)$ and $i(N)$, so it remains to prove that $\operatorname{Hom}_A(i(M), i(N))$ is constant. An explicit computation shows that the degeneracy map $\mathsf{Mod}_{\pi_0(A)}(M,N) \to \mathsf{sMod}_A(\Delta^n \otimes i(M), i(N))$ is just the adjunction isomorphism under the identification $M \cong \pi_0(\Delta^n \otimes i(M))$. It follows by monoidal nonsense that the canonical map

$$\pi_0(M \otimes_A i(N)) \to \pi_0(M) \otimes_{\pi_0(A)} N$$

is always an isomorphism. Since $M \otimes^{\mathbf{L}}_A i(N) = QM \otimes_A i(N)$, we find an isomorphism

$$\mathbf{L}\pi_0(M \otimes^{\mathbf{L}}_A i(N)) \cong \pi_0(M) \otimes_{\pi_0(A)} N. \tag{16}$$

Let $A \in \mathsf{sComm}_k$ and let M be a simplicial A-module. We call M *strong* if the induced map

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_*(M)$$

is an isomorphism. A morphism $A \to B$ in sComm_k is called *strong* if B is a strong A-module.

Lemma 17. *Let A be a simplicial commutative k-algebra and let M and N be simplicial A-modules such that N is strong and $\pi_0(N)$ is a flat $\pi_0(A)$-module. Then the natural map*

$$\pi_*(M) \otimes_{\pi_0(M)} \pi_0(N) \to \pi_*(M \otimes^{\mathbf{L}}_A N)$$

is an isomorphism.

Proof. This follows from the Künneth spectral sequence

$$E^2_{pq} = \operatorname{Tor}^{\pi_*(A)}_p(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_{p+q}(M \otimes^{\mathbf{L}}_A N)$$

of [**Qui67**, §6]. Since N is strong, we have

$$? \otimes_{\pi_*(A)} \pi_*(N) \cong ? \otimes_{\pi_*(A)} (\pi_*(A) \otimes_{\pi_0(A)} \pi_0(N)) \cong ? \otimes_{\pi_0(A)} \pi_0(N)$$

and so the flatness of $\pi_0(N)$ over $\pi_0(A)$ implies the flatness of $\pi_*(N)$ over $\pi_*(A)$. Therefore $E^2_{pq} = 0$ unless $p = 0$ and we obtain the required isomorphism $E^2_{0*} \cong \pi_*(M \otimes^{\mathbf{L}}_A N)$. □

Under the hypotheses of the lemma, we say that N is *flat* over A.

Corollary 18. *Strong morphisms are stable under composition and derived base change.*

Proof. Composition follows directly from the definition and base change is a consequence of the lemma. □

A morphism $A \to B$ in sComm_k is called *flat* (resp. *unramified*; *étale*) if it is strong and the induced morphism $\pi_0(A) \to \pi_0(B)$ of commutative k-algebras is flat (resp. unramified; étale). Recall that a map $A \to B$ of k-algebras is unramified if B is of finite type over A and if the B-module of differentials $\Omega_{B/A}$ is zero, and that it is étale if it is both flat and unramified. Thus a morphism in sComm_k is étale if and only if it is flat and unramified. Since flat and unramified morphisms in Comm_k are stable under compositions and base change, we obtain using Lemma 17 and its corollary that flat, unramified, and hence étale morphisms are all stable under composition and derived base change.

We put $\mathsf{dAff}_k = \mathsf{sComm}_k^{\mathrm{op}}$ and we endow dAff_k with the "opposite" model structure. When we think of a simplicial algebra A as an object in dAff_k we often denote it by $\operatorname{Spec} A$ instead. As the model category dAff_k is a simplicial model category, we shall always use the canonical cosimplicial and simplicial resolution functors in applying the definitions of Chapter 2, and, since all objects of dAff_k are cofibrant, we use the identity functor as cofibrant replacement functor. For instance, the functor $\underline{h}\colon \mathsf{dAff}_k \to \mathsf{dAff}_k^\wedge$ is defined by

$$\underline{h}_{\operatorname{Spec} A}(B) = \mathsf{sComm}_k(A, \Gamma_*(B)) = \operatorname{Map}(A, B),$$

where Map is the simplicial hom set of sComm_k.

We shall endow dAff_k with two model topologies. A family of maps $\{A \to B_i\}_{i\in I}$ in sComm_k is called a *flat covering* (resp. an *étale covering*) (of $\operatorname{Spec} A$) if

- each morphism $A \to B_i$ is flat (resp. étale);
- there exists a finite subset $J \subset I$ such that every prime ideal in $\pi_0(A)$ is the preimage of a prime ideal in $\prod_{i\in J} \pi_0(B_i)$.

These are equivalent to the conditions

- each morphism $A \to B_i$ is strong;
- $\{\pi_0(A) \to \pi_0(B_i)\}_{i\in I}$ is a flat covering (resp. an étale covering) in Comm_k.

In both cases, $\{\pi_0(A) \to \pi_0(B_i)\}_{i\in I}$ is a flat covering meaning that the family of base extension functors $\{\pi_0(f_i)_*\colon \mathsf{Mod}_{\pi_0(A)} \to \mathsf{Mod}_{\pi_0(B_i)}\}_i$ preserves and detects exact sequences (and in particular isomorphisms). In general we shall say that a family of functors is *conservative* if it detects isomorphisms.

Lemma 19. *Let $\{f_i\colon A \to B_i\}_i$ be a flat covering in sComm_k. Then the family of derived base change functors $\{\mathbf{L}(f_i)_*\colon \operatorname{Ho}\mathsf{sMod}_A \to \operatorname{Ho}\mathsf{sMod}_{B_i}\}_i$ is conservative.*

Proof. It suffices to prove that $\{(f_i)_*\}_i$ detects weak equivalences between cofibrant objects. Suppose that M and N are cofibrant and that $M \to N$ induces weak equivalences $M \otimes_A B_i \to N \otimes_A B_i$ for all i, i.e., it induces isomorphisms

$$\pi_*(M \otimes_A B_i) \cong \pi_*(N \otimes_A B_i)$$

for all i or equivalently, by Lemma 17, isomorphisms

$$\pi_*(M) \otimes_{\pi_0(A)} \pi_0(B_i) \cong \pi_*(N) \otimes_{\pi_0(A)} \pi_0(B_i).$$

But $\{\pi_0(f_i)_*\}_i$ is conservative, and so $M \to N$ induces isomorphisms $\pi_*(M) \to \pi_*(N)$. □

One defines a model topology on dAff_k, called the *flat topology*, as follows: a sieve S over x is a covering sieve if and only if it is generated by the image in $\operatorname{Ho}\mathsf{dAff}_k$ of a flat covering of x. It will be denoted by fl. We also define in the obvious way the *étale topology*, denoted by ét.

Proposition 20. fl *and* ét *are model topologies on* dAff_k*.*

Proof. This is an easy consequence of the analogous fact in the underived situation and the properties of strong morphisms. Let us prove for example that the stability axiom is satisfied (this is the only potentially nontrivial axiom to verify), say for the étale topology. Let S be the sieve generated by the image of an étale covering $\{A \to B_i\}_i$, and let $f\colon A \to A'$ be a morphism in $\mathrm{Ho}\,\mathsf{sComm}_k$, represented by a morphism $QA \to A'$ in sComm_k. Using the same argument as in the proof of Lemma 14 (where we really proved the weak universality of homotopy pullbacks), we obtain that $f^*(S)$ is the sieve generated by the image in $\mathrm{Ho}\,\mathsf{dAff}_k$ of the family of morphisms $\{A' \to A' \otimes^{\mathbf{L}}_{QA} B_i\}$ which are strong by Corollary 18. By Lemma 17, the π_0 of this family is $\{\pi_0(A') \to \pi_0(A') \otimes_{\pi_0(A)} \pi_0(B_i)\}$ which is an étale covering in Comm_k since étale coverings generate a classical topology on $\mathsf{Comm}_k^{\mathrm{op}}$. □

It is clear from the definitions that these model topologies are quasi-compact, and Proposition 16 is therefore applicable.

Theorem 21. *Let τ denote either the flat or the étale topology. Let $(X_i)_{i\in I}$ be a family of objects in dAff_k with I finite. Then the canonical map*

$$\coprod_{i\in I}^{\mathbf{L}} \mathbf{R}\underline{h}_{X_i} \to \mathbf{R}\underline{h}_{\amalg^{\mathbf{L}}_{i\in I} X_i}$$

is an equivalence in $\mathsf{dAff}_k^{\sim,\tau}$.

Proof. By induction it suffices to prove the lemma for I empty or with exactly two elements. If I is empty then the claim is that the unique map $\emptyset \to \mathbf{R}\underline{h}_0$ is a τ-local equivalence, where $\emptyset$ is the constant functor $\mathsf{sComm}_k \to \mathsf{sSet}$ with value the empty set and 0 is the zero algebra. Now the empty family is clearly a covering family of $0 \in \mathsf{dAff}_k$, so if $C_* \to \mathbf{R}\underline{h}_0$ denotes its homotopy nerve, $\underrightarrow{\mathrm{holim}}\, C_* \to \mathbf{R}\underline{h}_0$ is a τ-local equivalence. But $C_n = \emptyset$ for all n (it is an empty coproduct) and hence $\underrightarrow{\mathrm{holim}}\, C_* = \emptyset$. [Note that $\emptyset \to \mathbf{R}\underline{h}_0$ is an isomorphism on every object of sComm_k except on the zero algebra where $\mathbf{R}\underline{h}_0(0) = \Delta^0$; the situation should be compared with that of the empty presheaf of sets on a topological space, or on any site in which the empty family covers the initial object, whose associated sheaf is everywhere empty except on the initial object where its value becomes the one-point set.]

It remains to prove that

$$\mathbf{R}\underline{h}_X \amalg^{\mathbf{L}} \mathbf{R}\underline{h}_Y \to \mathbf{R}\underline{h}_{X\amalg^{\mathbf{L}} Y} = \mathbf{R}\underline{h}_{X\amalg Y}$$

is a τ-local equivalence. We again use the obvious fact the the family $\{X \to X \amalg Y, Y \to X \amalg Y\}$ is a τ-covering family. Its homotopy nerve $C_* \to \mathbf{R}\underline{h}_{X\amalg Y}$ is

$$\cdots \substack{\to\\\to\\\to} \mathbf{R}\underline{h}(X \times^{\mathbf{R}}_{X\amalg Y} X) \amalg^{\mathbf{L}} \mathbf{R}\underline{h}(X \times^{\mathbf{R}}_{X\amalg Y} Y) \amalg^{\mathbf{L}} \mathbf{R}\underline{h}(Y \times^{\mathbf{R}}_{X\amalg Y} X) \amalg^{\mathbf{L}} \mathbf{R}\underline{h}(Y \times^{\mathbf{R}}_{X\amalg Y} Y)$$
$$\rightrightarrows \mathbf{R}\underline{h}_X \amalg^{\mathbf{L}} \mathbf{R}\underline{h}_Y \to \mathbf{R}\underline{h}_{X\amalg Y}.$$

We will prove that C_* is levelwise equivalent in $\mathsf{dAff}_k^{\sim,\tau}$ to the constant simplicial object C_0. It will follow that $\underrightarrow{\mathrm{holim}}\, C_* \cong \underrightarrow{\mathrm{holim}}\, C_0$ in $\mathsf{dAff}_k^{\sim,\tau}$; since the colimit functor and the constant functor form a Quillen adjunction for the Reedy structure (see [**Hir03**, §15.10]), we have $\underrightarrow{\mathrm{holim}}\, C_0 = C_0$ and the proof will be complete. Write $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$. We shall prove below that

$$A \otimes^{\mathbf{L}}_{A\times B} B \cong 0. \tag{17}$$

In C_n there is one term which is an $(n+1)$-fold derived tensor product of A over $A \times B$, one which is an $(n+1)$-fold derived tensor product of B over $A \times B$, and all the other derived tensor products have a factor of the form $A \otimes^{\mathbf{L}}_{A\times B} B \cong 0$. Recall that the underlying simplicial k-module of a homotopy pushout $A \otimes^{\mathbf{L}}_C B$ of simplicial commutative k-algebras is also the underlying k-module of the derived tensor product of A and B in sMod_C; this implies $0 \otimes^{\mathbf{L}}_{A\times B} C \cong 0$ for any C and so all these mixed tensor products vanish. Using the first part of the proof we obtain that $C_* \to \mathbf{R}\underline{h}_x$ is levelwise τ-locally equivalent to

$$\cdots \substack{\to\\\to\\\to} \mathbf{R}\underline{h}(X \times^{\mathbf{R}}_{X\amalg Y} X) \amalg^{\mathbf{L}} \mathbf{R}\underline{h}(Y \times^{\mathbf{R}}_{X\amalg Y} Y) \rightrightarrows \mathbf{R}\underline{h}_X \amalg^{\mathbf{L}} \mathbf{R}\underline{h}_Y \to \mathbf{R}\underline{h}_{X\amalg Y}.$$

More precisely, we have proved that the obvious inclusion of this simplicial object into C_* is a levelwise τ-local equivalence. We can map the constant simplicial object C_0 into the above simplicial object diagonally, and we claim that this is a levelwise equivalence in $\mathsf{dAff}_k^\wedge$. For this it suffices to prove that the folding map induces an equivalence

$$A \otimes^{\mathbf{L}}_{A\times B} A \cong A. \tag{18}$$

We now prove (17) and (18). First we note that A and B are strong over $A \times B$ since

$$\pi_*(A\times B) \otimes_{\pi_0(A\times B)} \pi_0(A) \cong (\pi_*(A) \times \pi_*(B)) \otimes_{\pi_0(A)\times\pi_0(B)} \pi_0(A) \cong \pi_*(A),$$

and $\pi_0(A)$ and $\pi_0(B)$ are flat over $\pi_0(A\times B) \cong \pi_0(A) \times \pi_0(B)$ (they are localizations of the latter). Lemma 17 then tells us that

$$\pi_*(A \otimes^{\mathbf{L}}_{A\times B} B) \cong \pi_*(A) \otimes_{\pi_0(A)\times\pi_0(B)} \pi_0(B) = 0$$

and that the map $\pi_*(A \otimes^{\mathbf{L}}_{A\times B} A) \to \pi_*(A)$ is identified with the map

$$\pi_*(A) \otimes_{\pi_0(A)\times\pi_0(B)} \pi_0(A) \to \pi_*(A)$$

which is clearly an isomorphism. □

Proposition 22. *Let τ be either the flat or the étale topology. The model category $\mathsf{dAff}_k^{\sim,\tau}$ is the left Bousfield localization of $\mathsf{dAff}_k^\wedge$ along the morphisms*

$$\underrightarrow{\operatorname{holim}}\, \mathbf{R}\underline{h}_{Y_*} \to \mathbf{R}\underline{h}_X \quad \text{and} \quad \coprod_{i\in I}^{\mathbf{L}} \mathbf{R}\underline{h}_{Z_i} \to \mathbf{R}\underline{h}_{\amalg^{\mathbf{L}}_{i\in I} Z_i},$$

where $\mathbf{R}\underline{h}_{Y_*} \to \mathbf{R}\underline{h}_X$ is a τ-hypercover and $\{Z_i\}_{i\in I}$ is a finite family of objects in dAff_k.

Proof. Let H_1 and H_2 denote these two sets of morphisms and let H be the set of all morphisms of the form $\underrightarrow{\operatorname{holim}}\, C_* \to \mathbf{R}\underline{h}_x$ for $C_* \to \mathbf{R}\underline{h}_x$ a finite hypercover. By Proposition 16, $\mathsf{dAff}_k^{\sim,\tau}$ is the left Bousfield localization of $\mathsf{dAff}_k^\wedge$ with respect to H. By Theorem 21, H_1 and H_2 are τ-local equivalences, so it suffices to prove that an $(H_1 \cup H_2)$-local object in $\mathsf{dAff}_k^\wedge$ is H-local. Let F be an $(H_1\cup H_2)$-local object, and let $C_* \to \mathbf{R}\underline{h}_x$ be an arbitrary finite hypercover which in degree n is

$$C_n = \coprod_{i\in I_n}^{\mathbf{L}} \mathbf{R}\underline{h}_{y_i}.$$

Recall that the face and degeneracy maps are induced by morphisms in dAff_k between the various y_i's. Let $C'_* \to \mathbf{R}\underline{h}_x$ be the augmented simplicial object which in degree n is

$$C'_n = \mathbf{R}\underline{h}_{\amalg^{\mathbf{L}}_{i\in I_n} y_i}$$

and with face and degeneracy maps induced by those of C_*. [**To prove:** $C'_* \to \mathbf{R}\underline{h}_x$ is a hypercover.] Then $\underrightarrow{\operatorname{holim}}\, C'_* \to \mathbf{R}\underline{h}_x$ belongs to H_1 and there is a morphism of hypercovers $C_* \to C'_*$ which belongs to H_2 at each level. Using H_1 and H_2-locality we find

$$\mathbf{R}\operatorname{Map}(\mathbf{R}\underline{h}_x, F) \cong \underleftarrow{\operatorname{holim}}\, \mathbf{R}\operatorname{Map}(C'_*, F) \cong \underleftarrow{\operatorname{holim}}\, \mathbf{R}\operatorname{Map}(C_*, F).$$ □

Using the derived Yoneda lemma one can rephrase Proposition 22 as follows. Recall that a prestack on dAff_k is an equivalence-preserving functor $\mathsf{sComm}_k \to \mathsf{sSet}$.

Corollary 23. *Let τ be either the flat or the étale topology and let F be a prestack on dAff_k. Then F is a stack if and only if*

- *for every τ-hypercover $Y_* \to X$ in dAff_k, $F(X) \to \underleftarrow{\operatorname{holim}}\, F(Y_*)$ is an equivalence of simplicial sets;*

- *for every finite family* $(Z_i)_i$ *in* dAff_k, $F(\coprod_i Z_i) \to \prod_i F(Z_i)$ *is an equivalence of simplicial sets.*

The *model category of derived stack* is $\mathsf{dSt}_k = \mathsf{dAff}_k^{\sim,\text{ét}}$. Its homotopy category can be identified with the full subcategory of $\operatorname{Ho}\mathsf{sSet}^{\mathsf{sComm}_k}$ consisting of equivalence-preserving functors having *étale hyperdescent*; such functors are called *derived stacks*. An object $X \in \mathsf{dSt}_k$ is a derived stack if and only if it is pointwise equivalent to a fibrant object in dSt_k.

In the remaining of this section we shall give a proof of the most basic result in the theory of derived stack which is the derived analogue to the faithfully flat descent theorem for affine schemes. It characterizes the "gluing data" necessary to define a module locally on a flat hypercover. As in all our proofs so far it will be proved by reduction to the known situation of commutative k-algebras. As a consequence we shall deduce that the flat and étale topologies are subcanonical. We recall first some results about (nonsimplicial) commutative k-algebras. To distinguish between our generalized hypercovers and the hypercovers in the context of presheaves of *sets* on a site, we call the latter Set-hypercovers. Let $\mathsf{Aff}_k = \mathsf{Comm}_k^{\text{op}}$. Faithfully flat descent for affine k-schemes can be formulated as follows. If $A \to B^*$ is an augmented cosimplicial object in Comm_k which is also a Set-hypercover for the flat topology, then the adjunction

$$? \otimes_A B^* \colon \mathsf{Mod}_A \rightleftarrows \mathsf{cMod}_{B^*} : \varprojlim = \pi^0$$

restricts to an equivalence between Mod_A and the full subcategory of cMod_{B^*} consisting of cartesian objects, where a cosimplicial B^*-module E^* is cartesian if for every $\phi\colon m \to n$ in Δ the induced map

$$E^m \otimes_{B^m} B^n \to E^n$$

is an isomorphism of B^n-modules.

A general result about representable Set-hypercovers on arbitrary ringed sites is that they can be used to compute cohomology by means of a spectral sequence (see [**AM69**, Cor. 8.15]). For the flat or étale site this has the following consequences. Let $A \to B^*$ be a Set-hypercover and let E be an A-module, corresponding to the cosimplicial B^*-module $E^* = E \otimes_A B^*$. There is a convergent spectral sequence

$$E_2^{pq} = \pi^p(H^q(B^*, E^*)) \Rightarrow H^{p+q}(A, E)$$

where $H^*(A, M)$ denotes the flat (resp. étale) cohomology of $\operatorname{Spec} A$ with values in a module M. This cohomology is known to vanish in positive degrees ([**Mil80**, III, 3.7 and 3.8]), so the spectral sequence says that $\pi^p(E^*) \cong H^p(A, E)$ which is zero unless $p = 0$, in which case it gives the already known isomorphism $\pi^0(E^*) \cong E$. In other words, the augmented cosimplicial module $E \to E^*$ is aspherical.

Finally, we prove that if $Y_* \to X$ is a flat (resp. étale) hypercover in dAff_k, then $\pi_0(Y_*) \to \pi_0(X)$ is a flat (resp. étale) Set-hypercover in Aff_k. By hypothesis, $\pi_0(Y_n) \to \pi_0(\mathbf{R}\operatorname{cosk}_{n-1}(Y_*)_n)$ is a covering map in the site Aff_k, so it suffices to check that

$$\operatorname{cosk}_{n-1}(\pi_0(Y_*))_n \cong \pi_0(\mathbf{R}\operatorname{cosk}_{n-1}(Y_*)_n).$$

We may assume that Y_* is Reedy fibrant since π_0 preserves equivalence. Then the formula follows from the fact that $\pi_0\colon \mathsf{dAff}_k \to \mathsf{Aff}_k$ is right adjoint and hence commutes with the formation of coskeletons.

We now consider the derived situation. Let $A \to B^*$ be an augmented cosimplicial object in sComm_k. We endow the category csMod_{B^*} of cosimplicial simplicial B^*-modules with the model structure for which equivalences and fibrations are defined pointwise. Extension and restriction of scalars give an adjunction

$$? \otimes_A B^* \colon \mathsf{csMod}_A \rightleftarrows \mathsf{csMod}_{B^*} : U,$$

which is a Quillen adjunction as the right adjoint preserves equivalences and fibrations. There is also an adjunction

$$i\colon \operatorname{Ho}\mathsf{sMod}_A \rightleftarrows \operatorname{Ho}\mathsf{csMod}_A : \varprojlim\text{holim}$$

by definition of the homotopy limit. Putting these together we obtain an adjunction

$$? \otimes_A^{\mathbf{L}} B^*\colon \operatorname{Ho} \mathsf{sMod}_A \rightleftarrows \operatorname{Ho} \mathsf{csMod}_{B^*} : \underleftarrow{\operatorname{holim}}\,. \tag{19}$$

An object E^* in csMod_{B^*} is called *cartesian* if for every morphism $\phi\colon m \to n$ in Δ the induced morphism

$$E^m \otimes_{B^m}^{\mathbf{L}} B^n \to E^n$$

is invertible in $\operatorname{Ho} \mathsf{sMod}_{B^n}$. Since $\otimes_A$ is left balanced, if M is a simplicial A-module, $M \otimes_A^{\mathbf{L}} B^* \cong QM \otimes_A B^*$. It follows that any object of the form $M \otimes_A^{\mathbf{L}} B^*$ is cartesian.

Theorem 24. *If $A \to B^*$ is a flat hypercover, the adjunction (19) restricts to an equivalence between* $\operatorname{Ho} \mathsf{sMod}_A$ *and the full subcategory of* $\operatorname{Ho} \mathsf{csMod}_{B^*}$ *consisting of cartesian objects.*

Proof. We first prove that the counit of the restricted adjunction is an isomorphism, i.e., that for any cartesian $E^* \in \mathsf{csMod}_{B^*}$ the map

$$(\underleftarrow{\operatorname{holim}}\, E^*) \otimes_A^{\mathbf{L}} B^* \to E^*$$

is a weak equivalence of cosimplicial simplicial B^*-modules. Since each $A \to B^m$ is flat, we have by Lemma 17

$$\pi_q((\underleftarrow{\operatorname{holim}}\, E^*) \otimes_A^{\mathbf{L}} B^*) \cong \pi_q(\underleftarrow{\operatorname{holim}}\, E^*) \otimes_{\pi_0(A)} \pi_0(B^*)$$

and so we must prove that the map

$$\pi_q(\underleftarrow{\operatorname{holim}}\, E^*) \otimes_{\pi_0(A)} \pi_0(B^*) \to \pi_q(E^*) \tag{20}$$

is an isomorphism, for all $q \geq 0$. To this end we will use the Bousfield–Kan spectral sequence

$$E_2^{pq} = \pi^p \pi_q(E^*) \Rightarrow \pi_{q-p}(\underleftarrow{\operatorname{holim}}\, E^*).$$

For any $\phi\colon m \to n$ in Δ the morphism $\phi_*\colon B^m \to B^n$ is flat and hence, by lemma 17,

$$\pi_q(E^m \otimes_{B^m}^{\mathbf{L}} B^n) \cong \pi_q(E^m) \otimes_{\pi_0(B^m)} \pi_0(B^n).$$

Since E^* is cartesian this means that the cosimplicial $\pi_0(B^*)$-module $\pi_q(E^*)$ is cartesian (in the underived sense). By faithfully flat descent for k-algebras, we obtain that

$$\underleftarrow{\lim}\, \pi_q(E^*) \otimes_{\pi_0(A)} \pi_0(B^*) \to \pi_q(E^*) \tag{21}$$

is an isomorphism for all $q \geq 0$ and that $\pi^p \pi_q(E^*) = 0$ if $p \neq 0$. This implies, on the one hand, that the Bousfield–Kan spectral sequence converges (by [**GJ99**, VI, Cor. 2.21]) and, on the other hand, that it collapses at E_2, showing that the canonical map

$$\pi_q(\underleftarrow{\operatorname{holim}}\, E^*) \to \underleftarrow{\lim}\, \pi_q(E^*) \tag{22}$$

is an isomorphism. By (21) and (22) we obtain that (20) is an isomorphism, as required.

It remains to prove that the unit is an isomorphism. For this it is enough to show that the left adjoint is conservative (by the triangular identities). It is clear that i is conservative. Let us prove that $? \otimes^{\mathbf{L}} B^*\colon \operatorname{Ho} \mathsf{csMod}_A \to \operatorname{Ho} \mathsf{csMod}_{B^*}$ is conservative. Let $f\colon M^* \to N^*$ be a morphism between cofibrant cosimplicial simplicial A-modules inducing a weak equivalence $M^* \otimes_A B^* \to N^* \otimes_A B^*$. By Lemma 19, each functor $? \otimes_A^{\mathbf{L}} B^m\colon \operatorname{Ho} \mathsf{sMod}_A \to \operatorname{Ho} \mathsf{sMod}_{B^m}$ is conservative, and so each f^m is an equivalence. By definition, this means that f is an equivalence. ☐

Corollary 25. *If $A \to B^*$ is a flat hypercover, then $A \to \underleftarrow{\operatorname{holim}}\, B^*$ is an equivalence.*

Proof. $A \to \underleftarrow{\operatorname{holim}}\, B^*$ is the unit of the equivalence of the theorem for the simplicial A-module A. ☐

Corollary 26. *The flat and étale topologies are subcanonical.*

Proof. Let $A \in \mathsf{sComm}_k$ and $W = \operatorname{Spec} A$. We must prove that $\mathbf{R}\underline{h}_W$ has flat (and therefore étale) hyperdescent. We use Corollary 23. Let $Y_* \to X$ be a representable flat hypercover. Then by Corollary 25 and the fact that $\mathbf{R}\underline{h}_W$ preserves equivalences,

$$\mathbf{R}\underline{h}_W(X) \cong \mathbf{R}\underline{h}_W(\underrightarrow{\operatorname{holim}}\, Y_*) \cong \mathbf{R}\operatorname{Map}(\underrightarrow{\operatorname{holim}}\, Y_*, W) \cong \underleftarrow{\operatorname{holim}}\, \mathbf{R}\operatorname{Map}(Y_*, W) \cong \underleftarrow{\operatorname{holim}}\, \mathbf{R}\underline{h}_W(Y_*).$$

If $(Z_i)_i$ is a family of objects in dAff_k, then

$$\mathbf{R}\underline{h}_W(\coprod_i Z_i) \cong \mathbf{R}\operatorname{Map}(\coprod_i Z_i, W) \cong \prod_i \mathbf{R}\operatorname{Map}(Z_i, W) \cong \prod_i \mathbf{R}\underline{h}_W(Z_i). \qquad \square$$

Thus, for any $A \in \mathsf{sComm}_k$, $\mathbf{R}\underline{h}_{\operatorname{Spec} A}$ is a stack for the flat and étale topologies; an object of dSt_k is called an *affine derived stack* if it is equivalent in dSt_k to a derived stack of the form $\mathbf{R}\underline{h}_{\operatorname{Spec} A}$. The derived Yoneda embedding induces an equivalence between the category $\operatorname{Ho}\mathsf{dAff}_k$ and the full subcategory of $\operatorname{Ho}\mathsf{dSt}_k$ consisting of affine derived stacks.

The *tautological stack* is $\mathbb{A}^1 = \underline{h}_{\operatorname{Spec} k[T]}$ where $k[T]$ is a constant simplicial algebra. As $k[T]$ is cofibrant and ét is subcanonical, $\mathbb{A}^1$ is indeed a derived stack. For $A \in \mathsf{sComm}_k$, since A is fibrant, $\mathbb{A}^1(A) = \operatorname{Map}(k[T], A) = A$. Thus, $\mathbb{A}^1$ is isomorphic to the forgetful functor $\mathsf{sComm}_k \to \mathsf{sSet}$. We let $\mathscr{O}$ be the contravariant simplicial functor represented by $\mathbb{A}^1$ on the simplicial category dSt_k: $\mathscr{O}(X) = \operatorname{Map}(X, \mathbb{A}^1)$; since $\mathbb{A}^1$ is a k-algebra object in dSt_k, $\mathscr{O}$ underlies a simplicial functor $\mathsf{dSt}_k \to \mathsf{dAff}_k$, which we still denote by $\mathscr{O}$. By the properties of mapping spaces $\mathscr{O}$ has a total right derived functor $\mathbf{L}\mathscr{O} = \mathbf{R}_{\text{ét}}\operatorname{Map}(?, \mathbb{A}^1)$ underlying a morphism of left $\operatorname{Ho}\mathsf{sSet}$-modules. Moreover, the composition $\mathbf{L}\mathscr{O}\mathbf{R}\underline{h}$ is isomorphic to the identity by the derived Yoneda lemma.

3.3 Derived versus underived

In this section we briefly compare classical "underived" stacks to derived stacks. The conclusion is that the homotopy theory of underived stacks is fully embedded into the homotopy theory of derived stacks, but that this embedding does not preserve the monoidal structure. We fix τ to be either the flat or étale model topology on dAff_k, and we also write τ for the classical flat or étale topology on $\mathsf{Aff}_k = \mathsf{Comm}_k^{\mathrm{op}}$. We endow Aff_k with the trivial model structure so that it becomes a model site with the topology τ. The model category $\mathsf{Aff}_k^{\sim,\tau}$ will be called the model category of *underived stacks*. The characterization of derived stacks given in Corollary 23 applies to underived stacks as well (the proof is indeed the same, except that all the arguments explicitly involving simplicial commutative k-algebras become simpler for commutative k-algebras).

The inclusion $i\colon \mathsf{Aff}_k \to \mathsf{dAff}_k$ is right adjoint to the evaluation at zero functor and left adjoint to the connected component functor $\pi_0\colon \mathsf{dAff}_k \to \mathsf{Aff}_k$. Since the latter obviously preserves fibrations and equivalences, (i, π_0) is a Quillen adjunction. The functor i induces an adjunction

$$i_!\colon \mathsf{sSet}^{\mathsf{Comm}_k} \rightleftarrows \mathsf{sSet}^{\mathsf{sComm}_k} : i^*,$$

where $i^*(F)(A) = F(i(A))$ and $i_!$ is given by left Kan extensions. This is a Quillen adjunction for the projective model structures since i^* preserves fibrations and equivalences. Moreover, since $i_!h = hi$ and i preserves equivalences, if $x \to y$ is an equivalence in C, $\mathbf{L}i_!$ sends $h_x \to h_y$ to an isomorphism in $\operatorname{Ho}\mathsf{dAff}_k^\wedge$. By the universal property of left Bousfield localization, we obtain a Quillen adjunction

$$i_!\colon \mathsf{Aff}_k^\wedge \rightleftarrows \mathsf{dAff}_k^\wedge : i^*. \tag{23}$$

The functor i^* also has a right adjoint, namely the functor

$$\pi_0^*\colon \mathsf{Aff}_k^\wedge \to \mathsf{dAff}_k^\wedge, \quad \pi_0^*(F)(A) = F(\pi_0(A)).$$

It is obvious that π_0^* preserves projective equivalences and fibrations, so it is right Quillen for the projective model structures. But the model structure on $\mathsf{Aff}_k^\wedge$ is just the projective model structure, and by universality we get a Quillen adjunction

$$i^*\colon \mathsf{dAff}_k^\wedge \rightleftarrows \mathsf{Aff}_k^\wedge : \pi_0^*. \tag{24}$$

Lemma 27. *The functor i preserves hypercovers.*

Proposition 28. *The adjunctions (23) and (24) are Quillen adjunctions between the model categories of stacks* $\mathsf{Aff}_k^{\sim,\tau}$ *and* $\mathsf{dAff}_k^{\sim,\tau}$.

Proof. To prove these statements it suffices, by the universal property of left Bousfield localization and [**Hir03**, 3.1.6], to prove that the right adjoints preserve fibrant objects. Let $F \in \mathsf{dAff}_k^{\sim,\tau}$ and $G \in \mathsf{Aff}_k^{\sim,\tau}$ be fibrant. We already know that $i^*(F)$ and $\pi_0^*(G)$ are fibrant in $\mathsf{Aff}_k^{\wedge}$ and $\mathsf{dAff}_k^{\wedge}$, respectively, so it remains to verify that $i^*(F)$ and $\pi_0^*(G)$ satisfy the two conditions of Corollary 23. Let $Y_* \to X$ be a representable hypercover on the model site (Aff_k, τ). By the lemma, $i(Y_*) \to i(X)$ is a hypercover on (dAff_k, τ); therefore

$$i^*(F)(X) = F(i(X)) \cong \underleftarrow{\mathrm{holim}}\, F(i(Y_*)) = \underleftarrow{\mathrm{holim}}\, i^*(F)(Y_*).$$

If $(Z_i)_i$ is a family of objects in Aff_k,

$$i^*(F)(\coprod_i Z_i) = F(i(\coprod_i Z_i)) \cong F(\coprod_i i(Z_i)) \cong \prod_i F(i(Z_i)) = \prod_i i^*(F)(Z_i).$$

Now let $Y_* \to X$ be a representable hypercover on dAff_k. Recall that $\pi_0(Y_*) \to \pi_0(X)$ is a hypercover, so that

$$\pi_0^*(F)(X) = F(\pi_0(X)) \cong \underleftarrow{\mathrm{holim}}\, F(\pi_0(Y_*)) = \underleftarrow{\mathrm{holim}}\, \pi_0^*(F)(Y_*).$$

Let $(Z_i)_i$ be a finite family of objects in dAff_k. Since π_0 preserves finite coproducts (i.e. finite products of simplicial algebras), we find

$$\pi_0^*(F)(\coprod_i Z_i) = F(\pi_0(\coprod_i Z_i)) \cong F(\coprod_i \pi_0(Z_i)) \cong \prod_i F(\pi_0(Z_i)) = \prod_i \pi_0^*(F)(Z_i). \qquad \square$$

Proposition 29. *The functor* $\mathbf{L}i_!\colon \mathrm{Ho}\,\mathsf{Aff}_k^{\sim,\text{ét}} \to \mathrm{Ho}\,\mathsf{dAff}_k^{\sim,\text{ét}}$ *is fully faithful.*

Proof. The derived left adjoint $\mathbf{L}i_!\colon \mathrm{Ho}\,\mathsf{Aff}_k^{\sim,\text{ét}} \to \mathrm{Ho}\,\mathsf{dAff}_k^{\sim,\text{ét}}$ is fully faithful if and only if for any $F\colon \mathsf{Comm}_k \to \mathsf{sSet}$ the unit $F \to \mathbf{R}i^*\mathbf{L}i_!(F)$ is an isomorphism. We prove this first when $F = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$ is an affine scheme. Since $i_! h = hi$ and since h_x is a cofibrant replacement of $\mathbf{R}\underline{h}_x$, the functor $\mathbf{L}i_!\mathbf{R}\underline{h}$ is induced by the equivalence-preserving functor hi. Thus, $\mathbf{L}i_!(F) = h_{\operatorname{Spec} i(A)}$. The canonical map $F(B) \to \mathbf{R}i^*\mathbf{L}i_!(F)(B)$ is then the composite

$$\mathsf{Comm}_k(A,B) \to \mathsf{sComm}_k(i(A), i(B)) = h_{\operatorname{Spec} i(A)}(i(B)) = \mathbf{R}i^*\mathbf{L}i_!(F)(B)$$

which is obviously an isomorphism. So the unit is an isomorphism in this case. An arbitrary $F \in \mathsf{Aff}_k^{\sim,\tau}$ may be written as a homotopy colimit of representables in $\mathsf{sSet}^{\mathsf{Comm}_k}$, and since the identity $\mathsf{sSet}^{\mathsf{Comm}_k} \to \mathsf{Aff}_k^{\sim,\tau}$ is left Quillen F is also a homotopy colimit of affine schemes in $\mathsf{Aff}_k^{\sim,\tau}$. A consequence of Proposition 28 is that $\mathbf{R}i^* = \mathbf{L}i^*$ is the derived functor of a left Quillen functor. Therefore $\mathbf{R}i^*\mathbf{L}i_!$ commutes with homotopy colimits and the general case is reduced to the affine case. $\square$

One can thus see any underived stack X (e.g. a scheme) as a derived stack $i(X)$. However, several constructions are not preserved by this embedding. For example, since the functor $\mathbf{R}\underline{h}$ commutes with homotopy limits, for affine underived stacks $X = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$, $Y = \mathbf{R}\underline{h}_{\operatorname{Spec} B}$, and $Z = \mathbf{R}\underline{h}_{\operatorname{Spec} C}$, one has $i(X \times_Z^{\mathbf{R}} Y) = \mathbf{R}\underline{h}_{\operatorname{Spec} i(A \otimes_C B)}$, while $i(X) \times_{i(Z)}^{\mathbf{R}} i(Y) = \mathbf{R}\underline{h}_{\operatorname{Spec}(i(A) \otimes_{i(C)}^{\mathbf{L}} i(B))}$.

3.4 Quasi-coherent modules and vector bundles

A map $f\colon A \to B$ between simplicial commutative k-algebras induces a functor $f_*\colon \mathsf{sMod}_A \to \mathsf{sMod}_B$ by extension of scalars, and if $g\colon B \to C$ is another morphism in sComm_k there is an isomorphism of functors $(gf)_* \cong g_* f_*$. We can make this isomorphism into an equality using the following well-known trick. We define a new category Qcoh_A as follows: an object (M, α) of Qcoh_A is the data of a simplicial B-module M_B for any $B \in (\mathsf{sComm}_k \downarrow A)$ and of isomorphisms

$\alpha_u\colon M_B \otimes_B C \to M_C$ for any morphism $u\colon B \to C$ in $(\mathsf{sComm}_k \downarrow A)$, subject to the condition that for any composable pair

$$B \xrightarrow{u} C \xrightarrow{v} D$$

in $(\mathsf{sComm}_k \downarrow A)$ one has the equality $\alpha_{vu} = \alpha_v(\alpha_u \otimes_C D)$. A morphism $\phi\colon (M,\alpha) \to (N,\beta)$ in Qcoh_A is a family of morphisms $\phi_B\colon M_B \to N_B$, $B \in (\mathsf{sComm}_k \downarrow A)$, such that $\phi_C\alpha_u = \beta_u(\phi_B \otimes_B C)$ for any morphism $u\colon B \to C$ over A. Then the projection $(M,\alpha) \mapsto M_A$, $\phi \mapsto \phi_A$, is an equivalence of categories $\mathsf{Qcoh}_A \to \mathsf{sMod}_A$. We can therefore put a model structure on Qcoh_A by defining a morphism to be an equivalence (resp. a fibration; a cofibration) if and only if its image in sMod_A is an equivalence (resp. a fibration; a cofibration). Let $f\colon A \to B$ be a morphism in sComm_k; it induces a functor $f_!\colon (\mathsf{sComm}_k \downarrow B) \to (\mathsf{sComm}_k \downarrow A)$. We define $f_*\colon \mathsf{Qcoh}_A \to \mathsf{Qcoh}_B$, $f_*(M,\alpha) = (f_*(M), f_*(\alpha))$, by

$$f_*(M)_C = M_{f_!(C)}, \quad f_*(\alpha)_u = \alpha_{f_!(u)}, \quad \text{and} \quad f_*(\phi)_C = \phi_{f_!(C)}.$$

Clearly there is now an equality $(gf)_* = g_*f_*$ for any composable pair (f,g) in sComm_k. Moreover, the diagram of categories

$$\begin{array}{ccc} \mathsf{Qcoh}_A & \xrightarrow{f_*} & \mathsf{Qcoh}_B \\ \downarrow & & \downarrow \\ \mathsf{sMod}_A & \xrightarrow[f_*]{} & \mathsf{sMod}_B \end{array}$$

commutes up to natural isomorphism, and so we have "strictified" our original lax functor $A \mapsto \mathsf{sMod}_A$. Since the bottom arrow in the above diagram is a left Quillen functor (its right adjoint preserves equivalences and fibrations), it follows from the definition of the model structures on Qcoh_A and Qcoh_B that the top arrow $f_*\colon \mathsf{Qcoh}_A \to \mathsf{Qcoh}_B$ is a left Quillen functor. In particular, it induces a functor

$$f_*\colon \mathsf{Qcoh}_A^{cw} \to \mathsf{Qcoh}_B^{cw}$$

between the categories of cofibrant objects and equivalences between them. Taking nerves we obtain a functor

$$\mathrm{Qcoh}\colon \mathsf{sComm}_k \to \mathsf{sSet}, \quad A \mapsto N(\mathsf{Qcoh}_A^{cw}).$$

Since the inclusion $\mathsf{Qcoh}_A^{cw} \subset \mathsf{Qcoh}_A^{w}$ is an equivalence of categories, it induces a homotopy equivalence $N(\mathsf{Qcoh}_A^{cw}) \to N(\mathsf{Qcoh}_A^{w})$. The object $\mathrm{Qcoh} \in \mathsf{dSt}_k$ is called the *derived stack of quasi-coherent modules*. It is proved in [**HAGII**, Thm. 1.3.7.2] that Qcoh is indeed a derived stack, i.e., that it preserves equivalences and has étale hyperdescent. That it preserves equivalences is an easy consequence of the fact that f_* is a Quillen equivalence when f is a weak equivalence. That it has hyperdescent is a direct consequence of Theorem 24, modulo a technical result ([**HAGII**, Cor. B.0.8]) that we do not reproduce here.

Let $A \in \mathsf{sComm}_k$. A simplicial A-module M is called *perfect* if

- it is strong and
- $\pi_0(M)$ is a finitely generated and projective $\pi_0(A)$-module.

Observe that perfect simplicial modules are flat. If $f\colon A \to B$ is a morphism of simplicial commutative k-algebras, the derived base change functor $? \otimes_A^{\mathbf{L}} B\colon \operatorname{Ho}\mathsf{sMod}_A \to \operatorname{Ho}\mathsf{sMod}_B$ preserves perfect modules. Indeed, by Lemma 17 we have

$$\pi_0(M \otimes_A^{\mathbf{L}} B) \cong \pi_0(M) \otimes_{\pi_0(A)} \pi_0(B)$$

which is an f.g.p. $\pi_0(B)$-module, and

$$\begin{aligned} \pi_0(M \otimes_A^{\mathbf{L}} B) \otimes_{\pi_0(B)} \otimes \pi_*(B) &\cong \pi_0(M) \otimes_{\pi_0(A)} \pi_0(B) \otimes_{\pi_0(B)} \pi_*(B) \\ &\cong \pi_0(M) \otimes_{\pi_0(A)} \pi_*(B) \cong \pi_*(M \otimes_A^{\mathbf{L}} B), \end{aligned}$$

so that $M \otimes_A^{\mathbf{L}} B$ is strong. In particular, the base change functor $f_*\colon \mathsf{sMod}_A \to \mathsf{sMod}_B$ preserves cofibrant perfect modules. We denote by Vect_A the full subcategory of Qcoh_A consisting of those objects whose image in sMod_A is perfect. Then we have a well-defined subfunctor Vect of Qcoh given by

$$\mathrm{Vect}\colon \mathsf{sComm}_k \to \mathsf{sSet}, \quad A \mapsto N(\mathsf{Vect}_A^{cw}),$$

where Vect_A^{cw} is the intersection of Vect_A and Qcoh_A^{cw}. Again, $N(\mathsf{Vect}_A^{cw})$ is homotopy equivalent to $N(\mathsf{Vect}_A^{w})$. It is proved in [**HAGII**, Cor. 1.3.7.4] that Vect is a derived stack (this follows from the fact that a simplicial module is perfect if and only if it is perfect étale-locally, which is readily proved by reduction to the underived case). It is called the *derived stack of vector bundles.*

In a closed symmetric monoidal category, an object x will be called *dualizable* when the canonical map

$$x \otimes x^\vee \to \mathrm{Hom}(x,x),$$

adjoint to

$$(x \otimes x^\vee) \otimes x \cong x \otimes (x^\vee \otimes x) \cong x \otimes (x \otimes x^\vee) \to x \otimes 1 \cong x,$$

is an isomorphism, where by definition $x^\vee = \mathrm{Hom}(x,1)$. If x is dualizable, then the natural map $x^\vee \otimes y \to \mathrm{Hom}(x,y)$ is an isomorphism for any y.

Lemma 30. *In a closed symmetric monoidal category* C*, a retract of a dualizable object is dualizable.*

Proof. Let y be dualizable and let the composition

$$x \xrightarrow{u} y \xrightarrow{v} x$$

be the identity. Then there is a commutative diagram

$$\begin{array}{ccccc} x \otimes x^\vee & \xrightarrow{u \otimes v^\vee} & y \otimes y^\vee & \xrightarrow{v \otimes u^\vee} & x \otimes x^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(x,x) & \xrightarrow[\mathrm{Hom}(v,u)]{} & \mathrm{Hom}(y,y) & \xrightarrow[\mathrm{Hom}(u,v)]{} & \mathrm{Hom}(x,x) \end{array}$$

whose rows are the identity. The lemma follows. ☐

Lemma 31. *Let $A \in \mathsf{sComm}_k$. Let M and N be simplicial A-modules such that N is a retract of M in* $\mathrm{Ho}\,\mathsf{sMod}_A$*. If M is a strong (resp. perfect), then N is strong (resp. perfect).*

Proof. This is clear. ☐

Lemma 32. *Let $A \in \mathsf{sComm}_k$. Let M and N be simplicial A-modules such that M is a retract of A^n in* $\mathrm{Ho}\,\mathsf{sMod}_A$ *for some $n \geq 0$. Then $\mathbf{L}\pi_0\colon [M,N] \to [\pi_0(M), \pi_0(N)]$ is a bijection.*

Proof. Since π_0 commutes with colimits, we may assume $M = A$. Note that $\mathsf{sMod}_A(A,N)$ is in bijection with N_0, an element $x \in N_0$ corresponding to the map f_x which in degree n is $a \mapsto as(x)$ where $s(x)$ is the degeneracy of x in degree n. Since A is cofibrant in sMod_A, $[A,N]$ is a set of homotopy classes. We claim that x and y become equal in $\pi_0(N)$ if and only if f_x and f_y are homotopic, i.e., if and only if there exists g in the diagram

$$\begin{array}{ccc} A \oplus A & \to & \Delta^1 \otimes A \\ & \searrow^{f_x + f_y} & \vdots\, g \\ & & N. \end{array}$$

Here $\Delta^1 \otimes A$ is defined using the sSet-module structure of sMod_A: in degree n it is a direct sum of $n+2$ copies of A_n. If g exists, let $z = g_1(0,1,0)$. Then $d_0(z) = x$ and $d_1(z) = y$. Conversely, suppose that there exists $z \in N_1$ such that $d_0(z) = x$ and $d_1(z) = y$. Let $s_1, \ldots, s_n$ be the n degeneracy maps $N_1 \to N_n$, s_i being induced by the surjective map $n \to 1$ with i zeros. Setting

$$g_n(a_0, \ldots, a_{n+1}) = s(x)a_0 + s_1(z)a_1 + \cdots + s_n(z)a_n + s(y)a_{n+1}$$

gives the required homotopy g. Thus, we obtain a bijection $[A, N] \cong \pi_0(N)$, $[f] \mapsto [f_0(1)]$; there is also a bijection $[\pi_0(A), \pi_0(N)] \cong \pi_0(N)$, $g \mapsto g([1])$. Since $\mathbf{L}\pi_0(f)([1]) = [f_0(1)]$, the map of the lemma is the composition of these two bijections. □

We recall some properties of Postnikov towers. If $M \in \mathsf{sMod}_A$, a Postnikov tower for M is any tower of simplicial A-modules under M

$$\cdots \to M_{\leq k} \to M_{\leq k-1} \to \cdots \to M_{\leq 1} \to M_{\leq 0}$$

such that

$$\pi_n(M_{\leq k}) = \begin{cases} \pi_n(M) & \text{if } n \leq k \\ 0 & \text{otherwise.} \end{cases}$$

If $\sim_k$ is the equivalence relation on M such that, for $x, y\colon \Delta^i \to M$ two i-simplices of M, $x \sim_k y$ if and only if the restriction of x and y to $\operatorname{sk}_k \Delta^i$ are equal, then the simplicial A-modules $M/{\sim_k}$ form a Postnikov tower. Any two Postnikov towers are pointwise equivalent. In fact, the simplicial A-module $M_{\leq k}$ is determined, up to equivalence, by the following universal property: it is *k-truncated*, i.e., $\pi_n(M_{\leq k}) = 0$ if $n > k$, and for any k-truncated simplicial A-module N, the map

$$\mathbf{R}\operatorname{Map}(M_{\leq k}, N) \to \mathbf{R}\operatorname{Map}(M, N)$$

is an isomorphism in $\operatorname{Ho}\mathsf{sSet}$. This is true for all $k \geq -1$ if we set $M_{\leq -1} = 0$. For any Postnikov tower, the homotopy fiber of the morphism $M_{\leq k} \to M_{\leq k-1}$ is an Eilenberg–Mac Lane simplicial A-module which has homotopy $\pi_k(M)$ concentrated in degree k. In other words, it is equivalent to $\Sigma^k i(\pi_k(M))$ where $\Sigma\colon \operatorname{Ho}\mathsf{sMod}_A \to \operatorname{Ho}\mathsf{sMod}_A$ is the suspension functor.

Lemma 33. *Let $A \in \mathsf{sComm}_k$ and $M, N \in \mathsf{sMod}_A$. For $0 \leq n \leq k$,*

$$\pi_n(M \otimes^{\mathbf{L}}_A N_{\leq k}) \cong \pi_n(M \otimes^{\mathbf{L}}_A N).$$

Proof. Let P be a k-truncated object in sMod_A. We claim that $\mathbf{R}\operatorname{Hom}_A(N, P)$ is k-truncated. As a simplicial set, $\mathbf{R}\operatorname{Hom}_A(N, P) = \mathbf{R}\operatorname{Map}(N, P)$, so by [**Hov99**, Lem. 6.1.2],

$$\pi_n(\mathbf{R}\operatorname{Hom}_A(N, P)) = \pi_n(\mathbf{R}\operatorname{Map}(N, P)) = [N, \Omega^n(P)].$$

But if $n > k$, $\Omega^n(P) \cong 0$, so $\pi_n(\mathbf{R}\operatorname{Hom}_A(N, P)) = 0$. Thus, if P is k-truncated,

$$\begin{aligned} \mathbf{R}\operatorname{Map}(M \otimes^{\mathbf{L}} N_{\leq k}, P) &\cong \mathbf{R}\operatorname{Map}(N_{\leq k}, \mathbf{R}\operatorname{Hom}_A(M, P)) \\ &\cong \mathbf{R}\operatorname{Map}(N, \mathbf{R}\operatorname{Hom}_A(M, P)) \cong \mathbf{R}\operatorname{Map}(M \otimes^{\mathbf{L}}_A N, P). \end{aligned}$$

This proves that $(M \otimes^{\mathbf{L}}_A N_{\leq k})_{\leq k} \cong (M \otimes^{\mathbf{L}}_A N)_{\leq k}$. □

Lemma 34. *Let I be a filtered index category. Then the functor $\varinjlim\colon \mathsf{sMod}_A^{\mathsf{I}} \to \mathsf{sMod}_A$ sends pointwise equivalences to equivalences.*

Proof. This follows from [**Hov99**, Lem. 7.4.1] and the fact the all objects in sMod_A are fibrant. □

The following lemma is Sous-lemme 3 in [**Toë06a**] and a detailed proof can be found there. The proof of Lemma 36 is also adapted from a similar result in [**Toë06a**].

Lemma 35. *Let C be a model category and let $\mathbb{N}$ be the poset of natural numbers. The canonical functor*

$$\operatorname{Ho}(\mathsf{C}^{\mathbb{N}}) \to \operatorname{Ho}(\mathsf{C})^{\mathbb{N}}$$

is full, where the equivalences in $\mathsf{C}^{\mathbb{N}}$ are the pointwise equivalences.

Lemma 36. *Let M be a retract of A^n in $\operatorname{Ho}\mathsf{sMod}_A$ for some $n \geq 0$. Then any idempotent $p\colon M \to M$ in $\operatorname{Ho}\mathsf{sMod}_A$ splits.*

Proof. We may assume that M is cofibrant. Then p is represented by a morphism $q\colon M \to M$ in sMod_A. Let $N = \underrightarrow{\operatorname{holim}}\, Y$ where Y is the diagram

$$M \xrightarrow{q} M \xrightarrow{q} M \xrightarrow{q} \cdots .$$

Let also X be the constant $\mathbb{N}$-diagram at M, so that $M = \underrightarrow{\operatorname{holim}}\, X$. Define maps $u\colon X \to Y$ and $v\colon Y \to X$ in $(\operatorname{Ho}\mathsf{sMod}_A)^{\mathbb{N}}$ by $u_n = v_n = p$, for all $n \in \mathbb{N}$ (here we use that $p^2 = p$). By Lemma 35, u and v lift to maps u' and v' in $\operatorname{Ho}(\mathsf{sMod}_A^{\mathbb{N}})$. Define $s = \underrightarrow{\operatorname{holim}}\, u'$ and $j = \underrightarrow{\operatorname{holim}}\, v'$. Fix $n \geq 0$. The functor $\pi_n^{\mathbb{N}}\colon \mathsf{sMod}_A^{\mathbb{N}} \to \mathsf{sMod}_{\pi_0(A)}^{\mathbb{N}}$ preserves pointwise equivalences and therefore there is a well-defined functor $\operatorname{Ho}(\mathsf{sMod}_A^{\mathbb{N}}) \to \mathsf{sMod}_{\pi_0(A)}^{\mathbb{N}}$, which clearly factors as

$$\begin{array}{ccc}
\operatorname{Ho}(\mathsf{sMod}_A^{\mathbb{N}}) & \longrightarrow & \mathsf{Mod}_{\pi_0(A)}^{\mathbb{N}}. \\
\downarrow & \nearrow & \\
(\operatorname{Ho}\mathsf{sMod}_A)^{\mathbb{N}} & &
\end{array}$$

By Lemma 34 and the fact that π_n commutes with filtered colimits, all faces in the diagram

$$\begin{array}{ccccc}
\mathsf{sMod}_A^{\mathbb{N}} & & \xrightarrow{\underrightarrow{\lim}} & & \mathsf{sMod}_A \\
& \searrow^{\pi_n^{\mathbb{N}}} & & & & \searrow^{\pi_n} \\
\downarrow & & \mathsf{Mod}_{\pi_0(A)}^{\mathbb{N}} & \xrightarrow{\underrightarrow{\lim}} & \downarrow & & \mathsf{Mod}_{\pi_0(A)} \\
& \nearrow & & & & \nearrow \\
\operatorname{Ho}(\mathsf{sMod}_A^{\mathbb{N}}) & & \xrightarrow[\underrightarrow{\operatorname{holim}}]{} & & \operatorname{Ho}\mathsf{sMod}_A
\end{array}$$

are commutative up to natural isomorphism, except possibly the bottom parallelogram. But its commutativity follows from the commutativity of the other faces and the universality of $\mathsf{sMod}_A^{\mathbb{N}} \to \operatorname{Ho}(\mathsf{sMod}_A^{\mathbb{N}})$. It follows that $\pi_n(js) = \underrightarrow{\lim}\, \pi_n(vu) = \pi_n(p)\colon \pi_n(M) \to \pi_n(M)$ since $\pi_n(vu)$ is $\pi_n(p)$ in each degree. Hence $js = p$ by Lemma 32. Similarly, $\pi_n(sj)\colon \pi_n(N) \to \pi_n(N)$ is the result of applying the functor $\underrightarrow{\lim}$ to the morphism

$$\begin{array}{ccccccc}
\pi_n(M) & \xrightarrow{\pi_n(q)} & \pi_n(M) & \xrightarrow{\pi_n(q)} & \pi_n(M) & \xrightarrow{\pi_n(q)} & \cdots \\
\downarrow{\scriptstyle \pi_n(q)} & & \downarrow{\scriptstyle \pi_n(q)} & & \downarrow{\scriptstyle \pi_n(q)} & & \\
\pi_n(M) & \xrightarrow[\pi_n(q)]{} & \pi_n(M) & \xrightarrow[\pi_n(q)]{} & \pi_n(M) & \xrightarrow[\pi_n(q)]{} & \cdots
\end{array}$$

in $\mathsf{Mod}_{\pi_0(A)}^{\mathbb{N}}$, and this is clearly the identity, so $\pi_n(sj) = \mathrm{id}$. In particular, sj is an automorphism of N in $\operatorname{Ho}\mathsf{sMod}_A$. Setting $t = (sj)^{-1}s$ yields $tj = \mathrm{id}$ and $\pi_0(jt) = \pi_0(j)\pi_0(sj)^{-1}\pi_0(s) = \pi_0(js)$, so that, again by Lemma 32, $jt = js = p$. Thus, j and t form a splitting of p. ☐

Theorem 37. *Let $A \in \mathsf{sComm}_k$ and let $M \in \mathsf{sMod}_A$. The following are equivalent:*

1. *M is dualizable in $\operatorname{Ho}\mathsf{sMod}_A$;*
2. *M is perfect;*
3. *M is a retract of A^n in $\operatorname{Ho}\mathsf{sMod}_A$ for some $n \geq 0$.*

Proof. $1 \Rightarrow 2$. Let M be dualizable in $\operatorname{Ho}\mathsf{sMod}_A$. We must prove that M is strong and that $\pi_0(M)$ is an f.g.p. $\pi_0(A)$-module, or equivalently, that it is flat and finitely presented. Let $N \to P$ be an injective morphism of $\pi_0(A)$-modules. Since a map between constant simplicial k-modules is always a fibration,

$$0 \to i(N) \to i(P)$$

is a fiber sequence in $\mathrm{Ho}\,\mathsf{sMod}_A$. The functor $\mathrm{Hom}_A(Q(M^\vee), ?)$ is a right Quillen functor and hence its total derived functor $\mathbf{R}\,\mathrm{Hom}_A(M^\vee, ?)$ preserves fiber sequences. Thus

$$0 \to \mathbf{R}\,\mathrm{Hom}_A(M^\vee, i(N)) \to \mathbf{R}\,\mathrm{Hom}_A(M^\vee, i(P))$$

is a fiber sequence. Since M is dualizable, this sequence is isomorphic to

$$0 \to M \otimes_A^{\mathbf{L}} i(N) \to M \otimes_A^{\mathbf{L}} i(P),$$

and so we obtain a long exact sequence

$$\cdots \to \pi_n(M \otimes_A^{\mathbf{L}} i(N)) \to \pi_n(M \otimes_A^{\mathbf{L}} i(P)) \to 0 \to \cdots \to 0 \to \pi_0(M \otimes_A^{\mathbf{L}} i(N)) \to \pi_0(M \otimes_A^{\mathbf{L}} i(P)).$$

In view of (16), the last three terms are isomorphic to $0 \to \pi_0(M) \otimes_{\pi_0(A)} N \to \pi_0(M) \otimes_{\pi_0(A)} P$ and this shows that $\pi_0(M)$ is a flat $\pi_0(A)$-module. For $N = 0$, the long exact sequence says that $\pi_n(M \otimes_A^{\mathbf{L}} i(P)) = 0$ if $n \geq 1$. Thus the unit

$$M \otimes_A^{\mathbf{L}} i(P) \to i(\pi_0(M) \otimes_{\pi_0(A)} P)$$

is an isomorphism in $\mathrm{Ho}\,\mathsf{sMod}_A$. Since the functor $M \otimes_A ?$ is left Quillen its total derived functor commutes with the suspension, and we obtain

$$M \otimes_A^{\mathbf{L}} \Sigma^k i(P) \cong \Sigma^k i(\pi_0(M) \otimes_{\pi_0(A)} P) \tag{25}$$

for every $k \geq 0$. Now let Q be any simplicial A-module and let $k \geq 0$. The Postnikov tower of Q gives us a fiber sequence

$$\Sigma^k i(\pi_k(Q)) \to Q_{\leq k} \to Q_{\leq k-1},$$

and since M is dualizable we find as above that

$$M \otimes_A^{\mathbf{L}} \Sigma^k i(\pi_k(Q)) \to M \otimes_A^{\mathbf{L}} Q_{\leq k} \to M \otimes_A^{\mathbf{L}} Q_{\leq k-1}$$

is again a fiber sequence. By (25), the associated long exact sequence looks like

$$\begin{aligned} \cdots \to \pi_{n+1}(M \otimes_A^{\mathbf{L}} Q_{\leq k-1}) \to \pi_n \Sigma^k i(\pi_0(M) \otimes_{\pi_0(A)} \pi_k(Q)) \\ \to \pi_n(M \otimes_A^{\mathbf{L}} Q_{\leq k}) \to \pi_n(M \otimes_A^{\mathbf{L}} Q_{\leq k-1}) \to \cdots . \end{aligned}$$

We prove by induction on $k \geq -1$ that $\pi_n(M \otimes_A^{\mathbf{L}} Q_{\leq k}) = 0$ for all $n \geq k+1$. If $k = -1$, this is clear since $Q_{\leq -1} = 0$. Suppose $k \geq 0$. By induction hypothesis, the long exact sequence gives an isomorphism

$$\pi_n \Sigma^k i(\pi_0(M) \otimes_{\pi_0(A)} \pi_k(Q)) \cong \pi_n(M \otimes_A^{\mathbf{L}} Q_{\leq k})$$

for all $n \geq k$. If $n \geq k+1$, the left-hand side vanishes, so our claim is proved. For $n = k$, we obtain by Lemma 33 an isomorphism

$$\pi_0(M) \otimes_{\pi_0(A)} \pi_n(Q) \cong \pi_n(M \otimes_A^{\mathbf{L}} Q).$$

If we take $Q = A$, this proves that M is strong.

It remains to prove that $\pi_0(M)$ is finitely presented. Let I be a filtered index category and $d\colon \mathsf{I} \to \mathsf{Mod}_{\pi_0(A)}$ a diagram. By Lemma 34 there is an isomorphism $\varinjlim id \cong \underrightarrow{\mathrm{holim}}\, id$ in $\mathrm{Ho}\,\mathsf{sMod}_A$. Then there is a sequence of isomorphisms

$$\begin{aligned} \mathbf{R}\,\mathrm{Map}(\pi_0(M), \varinjlim d) \cong \mathbf{R}\,\mathrm{Map}(M, \varinjlim id) \cong \mathbf{R}\,\mathrm{Map}(M, \underrightarrow{\mathrm{holim}}\, id) \cong M^\vee \otimes_A^{\mathbf{L}} \underrightarrow{\mathrm{holim}}\, id \\ \cong \underrightarrow{\mathrm{holim}}(M^\vee \otimes_A^{\mathbf{L}} id) \cong \underrightarrow{\mathrm{holim}}\, \mathbf{R}\,\mathrm{Map}(M, id) \cong \underrightarrow{\mathrm{holim}}\, \mathbf{R}\,\mathrm{Map}(\pi_0(M), d) \end{aligned}$$

whose composition is clearly the canonical map

$$\mathsf{Mod}_{\pi_0(A)}(\pi_0(M), \varinjlim d) \to \varinjlim \mathsf{Mod}_{\pi_0(A)}(\pi_0(M), d).$$

Since any module is a filtered colimit of finitely presented modules, $\pi_0(M)$ itself is finitely presented.

$2 \Rightarrow 3$. We suppose that M is a perfect simplicial A-module. Choose a diagram

$$\pi_0(M) \xrightarrow{i} \pi_0(A^n) \xrightarrow{p} \pi_0(M)$$

such that $pi = \mathrm{id}$. By Lemma 32, p (resp. ip) is the image of a unique morphism $q\colon A^n \to M$ (resp. $r\colon A^n \to A^n$) in $\mathrm{Ho}\,\mathsf{sMod}_A$. Since $ipip = ip$ and $pip = p$, we have $r^2 = r$ and $qr = q$. By Lemma 36, the idempotent r splits: there exists a simplicial A-module N and maps $s\colon A^n \to N$ and $j\colon N \to A^n$ in the homotopy category such that $sj = \mathrm{id}$ and $js = r$. In particular N is a retract of A^n in $\mathrm{Ho}\,\mathsf{sMod}_A$. Then we have $\pi_0(qj)\pi_0(s)i = \pi_0(qr)i = \pi_0(q)i = pi = \mathrm{id}$ and $\pi_0(s)i\pi_0(qj) = \pi_0(s)ip\pi_0(j) = \pi_0(srj) = \pi_0(sjsj) = \mathrm{id}$, so that $\pi_0(qj)\colon \pi_0(N) \to \pi_0(M)$ is invertible. But N is strong by Lemma 31 (since A^n is clearly strong), and M is strong by hypothesis, so $\pi_*(qj)$ is an isomorphism, i.e., $qj\colon N \to M$ is an isomorphism. In particular M is also a retract of A^n in $\mathrm{Ho}\,\mathsf{sMod}_A$.

$3 \Rightarrow 1$. It is clear that, in a general setting, a biproduct of dualizable objects is dualizable. This implies that A^n is dualizable, and Lemma 30 completes the proof. □

Corollary 38. *Let $A \in \mathsf{sComm}_k$. The restriction of the functor $\mathbf{L}\pi_0\colon \mathrm{Ho}\,\mathsf{sMod}_A \to \mathsf{Mod}_{\pi_0(A)}$ to the full subcategory of dualizable objects is fully faithful.*

Proof. This follows from Lemma 32 and Theorem 37. □

We end this section with a very informal discussion of the categories of quasi-coherent modules and vector bundles on a derived stack. A quasi-coherent module on an affine derived stack should be the same thing as a simplicial module on the corresponding simplicial k-algebra, while a quasi-coherent module on an arbitrary derived stack X should be an object that restricts to a simplicial module on every affine derived stack over X. If we write X as a homotopy colimit $\underrightarrow{\mathrm{holim}}_i X_i$ of affine derived stacks, this means that a quasi-coherent module on X is constructed by glueing quasi-coherent modules over each X_i:

$$\{\text{quasi-coherent modules on } X\} \cong \underleftarrow{\lim}\{\text{quasi-coherent modules on } X_i\}.$$

By the derived Yoneda lemma, a quasi-coherent module on X_i is the same thing as a morphism $X_i \to \mathrm{Qcoh}$ in $\mathrm{Ho}\,\mathsf{dSt}_k$:

$$\{\text{quasi-coherent modules on } X_i\} \cong [X_i, \mathrm{Qcoh}],$$

and so

$$\{\text{quasi-coherent modules on } X\} \cong \underleftarrow{\lim}[X_i, \mathrm{Qcoh}] = [\underrightarrow{\mathrm{holim}}\, X_i, \mathrm{Qcoh}] = [X, \mathrm{Qcoh}].$$

Thus, whatever a quasi-coherent module on X really is, it should be the same thing as a map $X \to \mathrm{Qcoh}$ in $\mathrm{Ho}\,\mathsf{dSt}_k$. Similarly, vector bundles on derived stacks should be classified by Vect. It turns out that model categories of quasi-coherent modules and vector bundles can be defined (in essentially the same was as they are defined for underived stacks) and that they satisfy these requirements. Since we will not have this construction at out disposal in the sequel, we *define* an equivalence class of quasi-coherent modules (resp. of vector bundles) on an object $X \in \mathsf{dSt}_k$ to be an element of $[X, \mathrm{Qcoh}]$ (resp. $[X, \mathrm{Vect}]$).

4 Loop spaces

4.1 Alternative descriptions of Hochschild and cyclic homology

In Chapter 1 have defined HH, HC, HC^{per}, and HC^{-} for cyclic k-modules, but we can define more generally these functors on the category of *mixed complexes*. A mixed complex (M, b, B) over k is at the same time a ($\mathbb{Z}$-graded) chain complex (M, b) and a cochain complex (M, B) whose differentials satisfy the relation $bB + Bb = 0$. Equivalently, a mixed complex is a differential graded (dg) module over the dg k-algebra $k[\epsilon]$ which is by definition

$$\cdots \to 0 \to k \xrightarrow{0} k \to 0 \to \cdots ,$$

where the two k's are in degrees 1 and 0. We write ϵ for the 1 in degree 1. If M is a dg $k[\epsilon]$-module, then it is a mixed complex where the map b is the differential and the map B corresponds to the action of ϵ. There is a functor $\mathsf{Mod}_k^{\Lambda^{\mathrm{op}}} \to \mathsf{dgMod}_{k[\epsilon]}$ sending a cyclic k-module E to the (nonnegatively graded) mixed complex (M, b, B) where (M, b) is the complex associated to the underlying simplicial k-module of E and $B = (1-t)s_{-1}N$. We will prove that HH, HC, and HC^{-} all factor through $\mathsf{dgMod}_{k[\epsilon]}$. This point of view presents the advantage that both cyclic homology and negative cyclic homology arise naturally as simple derived functors. Before proving this we recall that for any dg algebra A over k, the category of left dg A-modules has a model structure in which a map is an equivalence (resp. a fibration) if the underlying map of (unbounded) complexes of k-modules is. In other words, an equivalence is a map that induces an isomorphism on homology, while a fibration is a map that is surjective in each degree. The shift automorphism $N \mapsto N[1]$ of dgMod_A is defined by $N[1]_n = N_{n+1}$, $d_{N[1]} = -d_N$, $a_{[1]}n = (-1)^{\deg a} an$ where the left-hand side is the scalar multiplication in $N[1]$ and the right-hand side is the scalar multiplication in N, and $f[1]_n = f_{n+1}$. To simplify we now assume that A is graded commutative. Then there are bifunctors

$$\otimes_A \colon \mathsf{dgMod}_A \times \mathsf{dgMod}_A \to \mathsf{dgMod}_A \quad \text{and} \quad \mathrm{Hom}_A \colon \mathsf{dgMod}_A^{\mathrm{op}} \times \mathsf{dgMod}_A \to \mathsf{dgMod}_A$$

defined as follows. For (M, d_M) and (N, d_N) two dg A-modules, their tensor product over A is as a graded k-module the usual tensor product of M and N over A (the identification is simply $am \otimes n = m \otimes an$), and its dg A-module structure is defined by

$$d(m \otimes n) = d_M m \otimes n + (-1)^{\deg m} m \otimes d_N n \quad \text{and} \quad a(m \otimes n) = am \otimes n.$$

In degree n, $\mathrm{Hom}_A(M, N)$ has the k-module of all A-module maps $M \to N[n]$, and for such a map f,

$$d(f) = d_N f - (-1)^n f d_M \quad \text{and} \quad (af)(m) = af(m)$$

(observe that we wrote d_N and a, not $d_{N[n]}$ or $a_{[n]}$). These sign conventions make dgMod_A into a closed symmetric monoidal category. In fact, dgMod_A is a symmetric monoidal model category. In particular, the bifunctors $\otimes_A$ and Hom_A have total derived functors $\otimes_A^{\mathbf{L}}$ and $\mathbf{R}\mathrm{Hom}_A$. Moreover, the tensor product is left balanced in the sense that $M \otimes_A ?$ preserves equivalences whenever M is cofibrant.

Lemma 39. *The mixed complex*

$$Qk = \cdots \leftrightarrows k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k \underset{\epsilon}{\overset{0}{\leftrightarrows}} k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k \leftrightarrows 0 \leftrightarrows \cdots$$

is cofibrant.

Proof. Consider a lifting problem

$$\begin{array}{ccc} & & M \\ & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle h} \\ Qk & \xrightarrow[f]{} & N \end{array}$$

where h is a trivial fibration in $\mathsf{dgMod}_{k[\epsilon]}$. We construct a lift g inductively. Put $g_n = 0$ for $n < 0$. Let $n \geq 0$. Suppose that we have defined g_i for all $i < n$ such that

1. $dg_i = g_{i-1}d$;
2. $\epsilon g_{i-1}(x) = g_i(\epsilon x)$ for all $x \in (Qk)_{i-1}$;
3. $h_i g_i = f_i$; and
4. if n is even, $\epsilon g_{n-1}(\epsilon) = 0$.

We consider two cases. Suppose that n is even (if $n = 0$, we write ϵ for $0 \in (Qk)_{-1}$). Since $f_{n-1}(\epsilon) = df_n(1)$, the class of $f_{n-1}(\epsilon)$ is zero in $H_{n-1}(N)$. But $f_{n-1}(\epsilon) = h_{n-1}g_{n-1}(\epsilon)$, and since h induces an isomorphism in homology, the class of $g_{n-1}(\epsilon)$ is zero in $H_{n-1}(M)$, i.e., there exists $x \in M_n$ such that $dx = g_{n-1}(\epsilon)$. Then $dh_n(x) = h_{n-1}(dx) = h_{n-1}g_{n-1}(\epsilon) = f_{n-1}(\epsilon) = f_{n-1}(d1) = df_n(1)$, and so the difference $h_n(x) - f_n(1)$ is in the kernel of d. Since h is an isomorphism in homology, there exists $y \in M_n$ such that $dy = 0$ and $h_n(y) - h_n(x) + f_n(1)$ is zero in $H_n(N)$; then there exists $z \in N_{n+1}$ such that $dz = h_n(y) - h_n(x) + f_n(1)$, and since h_{n+1} is surjective, there exists $w \in M_{n+1}$ such that $h_{n+1}(w) = z$. Put

$$g_n(1) = x - y + dw.$$

We must check that this satisfies 1–4:

1. $dg_n(1) = dx - dy = dx = g_{n-1}(\epsilon) = g_{n-1}(d1)$;
2. $\epsilon g_{n-1}(\epsilon) = 0 = g_n(0) = g_n(\epsilon^2)$;
3. $h_n g_n(1) = h_n(x) - h_n(y) + h_n(dw) = h_n(x) - h_n(y) + dz = f_n(1)$; and
4. does not apply.

If n is odd, define $g_n(\epsilon) = \epsilon g_{n-1}(1)$. We check 1–4:

1. $dg_n(\epsilon) = -\epsilon dg_{n-1}(1) = -\epsilon g_{n-2}(d1) = g_{n-1}(-\epsilon d1) = g_{n-1}(d\epsilon)$;
2. by definition;
3. $h_n g_n(\epsilon) = \epsilon h_{n-1}g_{n-1}(1) = \epsilon f_{n-1}(1) = f_n(\epsilon)$; and
4. $\epsilon g_n(\epsilon) = \epsilon^2 g_{n-1}(1) = 0$. □

The "*HC*" part of the next theorem was proved in [**Kas87**].

Theorem 40. *If M is a nonnegatively graded mixed complex, then $HC(M) = H(k \otimes^{\mathbf{L}}_{k[\epsilon]} M)$ and $HC^-(M) = H(\mathbf{R}\operatorname{Hom}_{k[\epsilon]}(k, M))$, where k is viewed as a dg $k[\epsilon]$-module concentrated in degree 0.*

Proof. It is clear that the map $Qk \to k$ which is the identity in degree 0 is an equivalence, so that, by the lemma, Qk is a cofibrant replacement of k in $\mathsf{dgMod}_{k[\epsilon]}$. Now,

$$(Qk \otimes_{k[\epsilon]} M)_n \cong M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots$$

because $\epsilon \otimes m = 1 \otimes \epsilon m$, and the differential is given by $d(1 \otimes m) = d1 \otimes m + 1 \otimes dm = 1 \otimes \epsilon m + 1 \otimes dm$. Thus there is an isomorphism between $Qk \otimes_{k[\epsilon]} M$ and $\operatorname{Tot}\mathscr{B}(M)$. Any map of $k[\epsilon]$-modules $f\colon Qk \to M[n]$ is entirely determined by its components f_{2k} that can be chosen arbitrarily, so we have

$$\operatorname{Hom}_{k[\epsilon]}(Qk, M)_n \cong M_n \times M_{n+2} \times M_{n+4} \times \cdots,$$

and the definition of the differential reduces to $d(m_n, m_{n+2}, m_{n+4}, \dots) = (dm_n, dm_{n+2} - \epsilon m_n, dm_{n+4} - \epsilon m_{n+2}, \dots)$. We find that $\operatorname{Hom}_{k[\epsilon]}(Qk, M)$ is isomorphic to the total complex of the bicomplex obtained from $\mathscr{B}^-(M)$ by changing the sign of the horizontal differentals; but this bicomplex is isomorphic to $\mathscr{B}^-(M)$, and so $\operatorname{Hom}_{k[\epsilon]}(Qk, M) \cong \operatorname{Tot}\mathscr{B}^-(M)$. We conclude by noting that $Qk \otimes_{k[\epsilon]}$? preserves equivalences, so that $k \otimes^{\mathbf{L}}_{k[\epsilon]} M = Qk \otimes_{k[\epsilon]} M$, and that all dg modules are fibrant, so that $\mathbf{R}\operatorname{Hom}_{k[\epsilon]}(k, M) = \operatorname{Hom}_{k[\epsilon]}(Qk, M)$. □

We will now explain how this theorem can be interpreted in a simplicial setting. We first show that the above theorem remains true if one replaces the category of unbounded dg $k[\epsilon]$-modules by that of nonnegatively differential graded $k[\epsilon]$-modules. Of course one can only hope to recover in this way the nonnegative part of negative cyclic homology, but this is the most interesting part since the negative part has period 2 and coincides with periodic cyclic homology. For a general nonnegatively differential graded k-algebra A, which we assume to be graded commutative, we denote by $\mathsf{dgMod}_A^{\geq 0}$ the category of nonnegatively differential graded A-modules. The inclusion

$$i\colon \mathsf{dgMod}_A^{\geq 0} \to \mathsf{dgMod}_A$$

has a right adjoint $\tau_{\geq 0}$ given by

$$\tau_{\geq 0}(M)_n = \begin{cases} Z_0(M) & \text{if } n = 0, \\ M_n & \text{otherwise.} \end{cases}$$

Observe that both i and $\tau_{\geq 0}$ preserve equivalences as they commute with the homology functors H_n for all $n \geq 0$. Since $\tau_{\geq 0}$ also preserves fibrations, this is a Quillen adjunction. The result that we have in mind is a formal consequence of this adjunction, but we first explore a more general situation since we shall use it again in the sequel.

Suppose that (F, G) is an adjunction between closed symmetric monoidal categories C and D together with a natural morphism

$$\nabla\colon G(X) \otimes G(Y) \to G(X \otimes Y).$$

From it we deduce by adjunction a natural morphism

$$F(X \otimes Y) \to F(X) \otimes F(Y), \tag{26}$$

namely the adjoint to $\nabla(\eta_X \otimes \eta_Y)$. There is also a natural map

$$G(\mathrm{Hom}(F(X), Y)) \to \mathrm{Hom}(X, G(Y)) \tag{27}$$

adjoint to the composition

$$\begin{aligned} G(\mathrm{Hom}(F(X), Y)) \otimes X \xrightarrow{\mathrm{id}\otimes\eta} & G(\mathrm{Hom}(F(X), Y)) \otimes GF(X) \\ & \xrightarrow{\nabla} G(\mathrm{Hom}(F(X), Y) \otimes F(X)) \to G(Y). \end{aligned}$$

Lemma 41. *(26) is an isomorphism if and only if (27) is an isomorphism.*

Proof. Suppose that (26) is an isomorphism. Then postcomposition by (27) is always an isomorphism as it is the composition

$$\begin{aligned} \mathsf{C}(Z, G(\mathrm{Hom}(F(X), Y))) &\cong \mathsf{D}(F(Z), \mathrm{Hom}(F(X), Y)) \cong \mathsf{D}(F(Z) \otimes F(X), Y) \\ &\cong \mathsf{D}(F(Z \otimes X), Y) \cong \mathsf{C}(Z \otimes X, G(Y)) \cong \mathsf{C}(Z, \mathrm{Hom}(X, G(Y))). \end{aligned}$$

The converse is proved in a similar way. □

Under the equivalent conditions of the lemma, if moreover the counit $FG(X) \to X$ is an isomorphism, we obtain that the natural map

$$G(\mathrm{Hom}(X, Y)) \to \mathrm{Hom}(G(X), G(Y))$$

is an isomorphism, while if the unit $Y \to GF(Y)$ is an isomorphism, there is a natural isomorphism

$$G(\mathrm{Hom}(F(X), F(Y))) \cong \mathrm{Hom}(X, Y)$$

In particular, if (F, G) is an equivalence of categories, then each of F and G commutes with both tensor products and internal hom's.

The adjunction $(i, \tau_{\geq 0})$ is a Quillen adjunction between closed monoidal model categories whose unit is an isomorphism and whose left adjoint is a monoidal functor. Applying Lemma 41 to the derived adjunction $(\mathbf{L}i, \mathbf{R}\tau_{\geq 0})$, we obtain a natural isomorphism

$$\mathbf{R}\tau_{\geq 0}\mathbf{R}\operatorname{Hom}_A(\mathbf{L}i(X), \mathbf{L}i(Y)) \cong \mathbf{R}\operatorname{Hom}_A(X, Y) \tag{28}$$

in $\operatorname{Ho}\mathsf{dgMod}_A^{\geq 0}$. In particular, for all $n \geq 0$ we have

$$H_n(\mathbf{R}\operatorname{Hom}_A(\mathbf{L}i(X), \mathbf{L}i(Y))) \cong H_n(\mathbf{R}\operatorname{Hom}_A(X, Y)),$$

and hence we obtain the following bounded version of theorem 40.

Corollary 42. *If* $M \in \mathsf{dgMod}_A^{\geq 0}$ *and if* $n \geq 0$*, then* $HC_n(M) = H_n(k \otimes^{\mathbf{L}}_{k[\epsilon]} M)$ *and* $HC_n^-(M) = H_n(\mathbf{R}\operatorname{Hom}_{k[\epsilon]}(k, M))$*, where* k *is viewed as a dg* $k[\epsilon]$*-module concentrated in degree* 0*.*

To obtain a simplicial version of theorem 40, we will now prove that the model category $\mathsf{dgMod}_{k[\epsilon]}^{\geq 0}$ is Quillen equivalent, through the normalization functor, to some model category of equivariant simplicial k-modules, and moreover that the derived equivalence preserves tensor products and internal hom's. For this we need some general results about monoid actions in enriched categories.

Let V be any of the following closed symmetric monoidal category: sMod_A, dgMod_A, or $\mathsf{dgMod}_A^{\geq 0}$. Let C be a V-module. For G a monoid in V and x an object of C, an *action of* G *on* x is a map of V-monoids $G \to \operatorname{Map}(x, x)$. By adjunction, a morphism $G \to \operatorname{Map}(x, x)$ in V is the same thing as either a map $\phi\colon x \to x^G$ or a map $\psi\colon G \otimes x \to x$. Then the property that $G \to \operatorname{Map}(x, x)$ is a monoid map translates to the commutativity of either one of the diagrams

$$\begin{array}{ccc} x & \longrightarrow & x^G \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi^G} \\ x^G & \xrightarrow[x^{\mu}]{} & (x^G)^G, \end{array} \qquad \begin{array}{ccc} G \otimes (G \otimes x) & \xrightarrow{G \otimes \psi} & G \otimes x \\ {\scriptstyle \mu \otimes x}\big\downarrow & & \big\downarrow \\ G \otimes x & \longrightarrow & x, \end{array} \tag{29}$$

where $\mu\colon G \otimes G \to G$ is the monoid structure of G and the associativity isomorphisms of the V-module structure have been used implicitely. Let x and y be two objects with an action of G. Then one defines the object $\operatorname{Map}_G(x, y)$ of G-equivariant maps from x to y as the equalizer

$$\operatorname{Map}_G(x, y) \to \operatorname{Map}(x, y) \rightrightarrows \operatorname{Map}(x, y^G),$$

where the two parallel maps come from the two possible ways to go from x to y^G, namely $x \to y \to y^G$ and $x \to x^G \to y^G$. Dually, one defines the G-equivariant tensor product $x \otimes_G y$ by the coequalizer diagram

$$G \otimes (x \otimes y) \rightrightarrows x \otimes y \to x \otimes_G y$$

where the two maps correspond to the action of G on either x or y. It can be seen that the objects $\operatorname{Map}_G(x, y)$ define a V-enriched category structure on the set of G-equivariant objects in C; it is denoted by C^G (and it can of course be identified with the category of V-enriched functors from G to C). The above equalizer diagram also defines a V-enriched forgetful functor $\mathsf{C}^G \to \mathsf{C}$ (which is actually the right adjoint of a monoidal adjunction, but we will not use this). We deduce directly from the definitions that C^G becomes a V-module with tensor and cotensor functors compatible with the forgetful functor $\mathsf{C}^G \to \mathsf{C}$ (i.e., $K \otimes x$ and x^K are defined as in C and endowed with the obvious G-actions).

Suppose now that G comes with an augmentation, i.e., a morphism of monoids $G \to 1$ where 1 is the unit in V. Then one can define the *fixed point functor* $\operatorname{fix}_G\colon \mathsf{C}^G \to \mathsf{C}$ by the equalizer diagram

$$\operatorname{fix}_G(x) \to x \rightrightarrows x^G$$

where the two maps are adjoint to the action $G \to \mathrm{Map}(x,x)$ and to the composition $G \to 1 \to \mathrm{Map}(x,x)$ pointing at the identity, respectively. Dually, the *orbit functor* $\mathrm{orb}_G\colon \mathsf{C}^G \to \mathsf{C}$ is defined by the coequalizer diagram

$$G \otimes x \rightrightarrows x \to \mathrm{orb}_G(x).$$

Since $\mathrm{Map}(x,?)\colon \mathsf{C} \to \mathsf{V}$ preserves limits, we see that fix_G is a V-enriched right adjoint to the "endow with the trivial action" functor $\mathsf{C} \to \mathsf{C}^G$. Dually, orb_G is a V-enriched left adjoint to this functor.

If G is a simplicial monoid, we put a simplicial model structure on sMod_k^G by declaring a map to be an equivalence (resp. a fibration) if the underlying map of simplicial k-modules is an equivalence (resp. a fibration). That this is a simplicial model structure follows from the following lemma.

Lemma 43. *There is an isomorphism of* sSet*-modules* $\mathsf{sMod}_k^G \cong \mathsf{sMod}_{k[G]}$ *where* $k[G]$ *is the simplicial monoid algebra of* G*. This isomorphism induces a bijection between equivalences and fibrations on both sides.*

Proof. We simply define the isomorphism on objects. The category sMod_k is enriched over itself in such a way the the underlying simplicial set of the internal hom $\mathrm{Hom}(X,Y)$ is $\mathrm{Map}(X,Y)$. Therefore a map of simplicial sets $G \to \mathrm{Map}(X,X)$ is equivalent to a map of simplicial k-modules $k[G] \to \mathrm{Hom}(X,X)$, which is in turn equivalent to a map of simplicial k-modules $k[G]\otimes_k X \to X$. Moreover, $G \to \mathrm{Map}(X,X)$ is a monoid map if and only if $k[G] \otimes_k X \to X$ is an action of the simplicial k-algebra $k[G]$ on X. □

Through this isomorphism we have $\mathrm{fix}_G = \mathrm{Hom}_{k[G]}(k,?)$ (resp. $\mathrm{orb}_G = k\otimes_{k[G]} ?$), because the two functors are defined as the equalizers (resp. coequalizers) of isomorphic diagrams. Similarly, if $\mathsf{C} = \mathsf{dgMod}_k$ enriched over itself and $A \to k$ an augmented commutative dg k-algebra, the category dgMod_k^A is isomorphic to dgMod_A, and through this isomorphism we have $\mathrm{fix}_A = \mathrm{Hom}_A(k,?)$ (resp. $\mathrm{orb}_A = k \otimes_A ?$). A similar conclusion holds for $\mathsf{C} = \mathsf{dgMod}_k^{\geq 0}$. In any of these situations, we define the *homotopy fixed point functor* $\mathbf{R}\,\mathrm{fix}_G\colon \mathrm{Ho}\,\mathsf{C}^G \to \mathrm{Ho}\,\mathsf{C}$ to be $\mathbf{R}\,\mathrm{Hom}_G(k,?)$, and dually the *homotopy orbit functor* $\mathbf{L}\,\mathrm{orb}_G$ is by definition $k \otimes_G^{\mathbf{L}} ?$. Since k is cofibrant in C, the natural transformations

$$\mathrm{fix}_G \to U \to \mathrm{orb}_G,$$

where $U\colon \mathsf{C}^G \to \mathsf{C}$ is the equivalence-preserving forgetful functor, induce natural transformations

$$\mathbf{R}\,\mathrm{fix}_G \to \mathbf{R}U = \mathbf{L}U \to \mathbf{L}\,\mathrm{orb}_G$$

derived from $\mathrm{Hom}_G(x,y) \to \mathrm{Hom}(x,y)$ and $x \otimes y \to x \otimes_G y$. When $A = k[\epsilon]$ we have

Proposition 44. *Let* $n \geq 0$*. The diagram of natural transformations*

$$HC_n^- \to HH_n \to HC_n$$

between functors $\mathrm{Ho}\,\mathsf{dgMod}_{k[\epsilon]}^{\geq 0} \to \mathsf{Mod}_k$ *described in §1.1 is isomorphic to the diagram*

$$H_n(\mathbf{R}\,\mathrm{fix}_{k[\epsilon]}) \to H_n \to H_n(\mathbf{L}\,\mathrm{orb}_{k[\epsilon]}).$$

Proof. Actually these two diagrams are already isomorphic at the level of the chain complexes before taking homology. With the notation of Lemma 39, the inclusion $k \to Qk$ is inverse to the projection $Qk \to k$ in the homotopy category. Comparing the proof of Theorem 40 and the definitions of §1.1, it is clear that the map $\mathrm{Tot}\,\mathscr{B}^0(M) \to \mathrm{Tot}\,\mathscr{B}(M)$ corresponds to the map $k \otimes_k M \to Qk \otimes_k M \to Qk \otimes_{k[\epsilon]} M$ which is $M \to \mathbf{L}\,\mathrm{orb}_{k[\epsilon]}(M)$, while the map $\mathrm{Tot}\,\mathscr{B}^-(M) \to \mathrm{Tot}\,\mathscr{B}^0(M)$ corresponds to $\mathrm{Hom}_{k[\epsilon]}(Qk,M) \to \mathrm{Hom}_k(Qk,M) \to \mathrm{Hom}_k(k,M)$ which is $\mathbf{R}\,\mathrm{fix}_{k[\epsilon]}(M) \to M$. □

Let A be a simplicial commutative k-algebra. By [**SS03**, Thm. 1.1 (2)], the normalization functor $N\colon \mathsf{sMod}_A \to \mathsf{dgMod}^{\geq 0}_{N(A)}$ is the right adjoint of a Quillen equivalence. Let Γ be its left adjoint. Note that N actually preserves and reflects equivalences and fibrations, as these are defined on the underlying k-module objects. The model categories on both sides are monoidal categories, and this adjunction is a weak monoidal adjunction for the shuffle map ∇ (which also defines the algebra structure on $N(A)$). Moreover, it is proved in [**SS03**, §4.4] that the morphism

$$\mathbf{L}\Gamma(X \otimes^{\mathbf{L}}_{N(A)} Y) \to \mathbf{L}\Gamma(X) \otimes^{\mathbf{L}}_{A} \mathbf{L}\Gamma(Y)$$

derived from the monoidal structure is an isomorphism in $\operatorname{Ho}\mathsf{sMod}_A$. Using Lemma 41, it follows that $(\mathbf{L}\Gamma, \mathbf{R}N)$ is an equivalence of nonunital closed monoidal categories (actually the units are preserved as well). In particular there is a natural isomorphism

$$\mathbf{R}N(\mathbf{R}\operatorname{Hom}_A(X,Y)) \to \mathbf{R}\operatorname{Hom}_{N(A)}(\mathbf{R}N(X), \mathbf{R}N(Y)).$$

If $A = k[G]$ and $X = k$, since $\mathbf{R}N(k) \cong k$, we obtain that the normalization functor preserves homotopy fixed points, i.e., that there is a canonical isomorphism

$$\mathbf{R}N\mathbf{R}\operatorname{fix}_G \cong \mathbf{R}\operatorname{fix}_{N(k[G])}\mathbf{R}N.$$

(Dually, *if* the canonical map $\mathbf{L}\Gamma(k) \to k$ is an isomorphism in $\operatorname{Ho}\mathsf{sMod}_{k[G]}$, the left adjoint $\mathbf{L}\Gamma$ preserves homotopy orbits; but I do not know if this is true in general.)

Let $S^1 = \Delta^1/\partial\Delta^1$ be the simplicial set obtained from Δ^1 by identifying the two endpoints at each level, and let $B\mathbb{Z}$ be the nerve of the abelian group $\mathbb{Z}$, which is a simplicial abelian group. It is well-known that the inclusion $S^1 \to B\mathbb{Z}$ sending $\partial\Delta^1$ to 0 and the other points of $(S^1)_n$ to the generators of $(B\mathbb{Z})_n = \mathbb{Z}^n$ is an equivalence of simplicial sets. Applying the equivalence-preserving functor $N(k[?])$ to this inclusion (here $k[?] = ? \otimes k$ is the left adjoint to the forgetful functor $\operatorname{Map}(k, ?)\colon \mathsf{sMod}_k \to \mathsf{sSet}$), we obtain an equivalence of simplicial k-modules $N(k[S^1]) \to N(k[B\mathbb{Z}])$. The normalization of $k[S^1]$ can be identified, as a dg k-module, to $k[\epsilon]$, by sending ϵ to $(-1,1) \in N(k[S^1])_2 \subset k[S^1]_2 = k^2$. The equivalence

$$f\colon k[\epsilon] \to N(k[B\mathbb{Z}])$$

is actually a morphism of dg k-algebras. This implies that the Quillen adjunction (f_*, f^*), where $f_*\colon \mathsf{dgMod}^{\geq 0}_{k[\epsilon]} \to \mathsf{dgMod}^{\geq 0}_{N(k[B\mathbb{Z}])}$ is the extension of scalars, is a Quillen equivalence. Since extension of scalars is a comonoidal functor, it follows from Lemma 41 that $(\mathbf{L}f_*, \mathbf{R}f^*)$ is an equivalence of nonunital closed monoidal categories (again, the units are in fact clearly preserved as well). In particular, the natural map

$$\mathbf{R}f^*(\mathbf{R}\operatorname{Hom}_{N(k[B\mathbb{Z}])}(X,Y)) \to \mathbf{R}\operatorname{Hom}_{k[\epsilon]}(\mathbf{R}f^*(X), \mathbf{R}f^*(Y)) \tag{30}$$

is an isomorphism. Since $\mathbf{R}f^*(k) = k$, $\mathbf{R}f^*$ also commutes with homotopy fixed points. We summarize what we proved in the next proposition.

Proposition 45. *There are two Quillen equivalences*

$$\Gamma\colon \mathsf{dgMod}^{\geq 0}_{N(k[B\mathbb{Z}])} \rightleftarrows \mathsf{sMod}^{B\mathbb{Z}}_k : N$$
$$f_*\colon \mathsf{dgMod}^{\geq 0}_{k[\epsilon]} \rightleftarrows \mathsf{dgMod}^{\geq 0}_{N(k[B\mathbb{Z}])} : f^*$$

whose derived equivalences preserve the closed monoidal structure. Moreover the derived right adjoints preserve homotopy fixed points.

4.2 The loop space

In Proposition 45 we have seen that the model categories $\mathsf{sMod}^{B\mathbb{Z}}_k$ of $B\mathbb{Z}$-equivariant simplicial k-modules and $\mathsf{dgMod}^{\geq 0}_{k[\epsilon]}$ of nonnegatively graded mixed complex are Quillen equivalent. The goal of this section is to prove that the Hochschild complex of an algebra is naturally an object in $\mathsf{sMod}_k^{B\mathbb{Z}}$ and that negative cyclic homology identifies with the homotopy fixed points of this object. We formulate these results from the geometric point of view of derived stacks.

The *loop space functor* $L\colon \mathsf{dSt}_k \to \mathsf{dSt}_k$ is the functor $?^{B\mathbb{Z}}$ coming from the sSet-module structure of dSt_k. This is a right Quillen functor (with left adjoint $B\mathbb{Z} \otimes ?$), and we have

$$\mathbf{R}L = ?^{\mathbf{R}B\mathbb{Z}}$$

in the notation of §2.1 (because simplicial sets are always cofibrant). In particular, the isomorphism $S^1 \cong B\mathbb{Z}$ in $\operatorname{Ho}\mathsf{sSet}$ induces an isomorphism $\mathbf{R}L \cong ?^{\mathbf{R}S^1}$.

Suppose that $X = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$ is an affine derived stack (A being a simplicial commutative k-algebra). Since $\mathbf{R}\underline{h}$ underlies a morphism of right $\operatorname{Ho}\mathsf{sSet}$-modules, there is a canonical isomorphism

$$\mathbf{R}L(X) \cong \mathbf{R}\underline{h}_{(\operatorname{Spec} A)^{\mathbf{R}S^1}} = \mathbf{R}\underline{h}_{\operatorname{Spec}(S^1 \otimes^{\mathbf{L}} A)}$$

in $\operatorname{Ho}\mathsf{dSt}_k$. In particular, $\mathbf{R}L(X)$ is an affine derived stack. This is closely related to Hochschild homology as follows. First note that when A is a commutative k-algebra, its Hochschild complex is endowed with a structure of simplicial commutative k-algebra. In this context the Hochschild complex is a functor $\mathsf{Comm}_k \to \mathsf{sComm}_k$, where Comm_k is the category of commutative k-algebra. Recall from (15) that for K a simplicial set, $K \otimes A$ is just the diagonal of the bisimplicial commutative k-algebra which in degree (p, q) is the coproduct

$$\bigotimes_{x \in K_p} A_q.$$

Theorem 46. *The Hochschild complex functor* $\mathsf{Comm}_k \to \mathsf{sComm}_k$ *is isomorphic to the restriction of the functor* $S^1 \otimes ?\colon \mathsf{sComm}_k \to \mathsf{sComm}_k$.

Proof. We must explicit the simplicial structure of S^1. In degree n, $(S^1)_n$ has $n+1$ elements x_n, y_n^1, ..., y_n^n: x_n is the 0-dimensional point and y_n^1, ..., y_n^n are the images of y_1^1 (the loop) by the n surjective maps $n \to 1$ in Δ. In particular,

$$(S^1 \otimes A)_n = A^{\otimes n+1},$$

so it remains to see that the face maps and degeneracy maps of S^1 induce those of the Hochschild complex. This is easy because they are induced by those in Δ^1: s_i misses y^i, an internal d_i collapses y^i and y^{i+1} to y^i, d_0 collapses x_n and y_n^1 to x_{n-1}, and d_n collapses y_n^n and x_n to x_{n-1}. This is exactly as required. □

We define the *topological Hochschild homology* of an arbitrary simplicial commutative k-algebra A to be the simplicial commutative k-algebra $S^1 \otimes^{\mathbf{L}} A$. This defines an endofunctor of $\operatorname{Ho}\mathsf{sComm}_k$. When A is a k-algebra, also viewed as a constant simplicial k-algebra, this definition thus yields, up to equivalence, the classical Hochschild complex whenever A is cofibrant in sComm_k, for instance when A is free. More generally, since S^1 is the homotopy pushout of Δ^0 and Δ^0 along $\partial\Delta^1$, for any $A \in \mathsf{sComm}_k$ we have

$$S^1 \otimes^{\mathbf{L}} A \cong A \otimes^{\mathbf{L}}_{A \otimes^{\mathbf{L}} A} A,$$

and we know that the coproducts on the right are also derived tensor products of the underlying simplicial k-modules; using this formula it is easy to prove that $S^1 \otimes^{\mathbf{L}} A$ and $S^1 \otimes A$ are equivalent whenever A is a flat commutative k-algebra (see [**Lod92**, Prop. 1.1.13]).

We have proved above that for $X = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$ an affine derived stack, $\mathbf{R}L(X)$ is the affine derived stack associated to the topological Hochschild homology of A. More generally, since $\mathbf{L}\mathscr{O}$ commutes with the left $\operatorname{Ho}\mathsf{sSet}$-module structures, we have:

Theorem 47. *Let* $X \in \mathsf{dSt}_k$. *Then* $\mathbf{L}\mathscr{O}\mathbf{R}L(X)$ *is naturally isomorphic to the topological Hochschild homology of* $\mathbf{L}\mathscr{O}(X)$.

Thus, Hochschild homology has a geometric interpretation as functions on the loop space. The next step is to relate negative cyclic homology to S^1-equivariant functions on the loop space.

The loop space LX is endowed with an action of the simplicial group $B\mathbb{Z}$, that is, a map of simplicial monoids $B\mathbb{Z} \to \mathrm{Map}(LX, LX)$. It is defined as the adjoint to the composition

$$LX = X^{B\mathbb{Z}} \to X^{(B\mathbb{Z}\times B\mathbb{Z})} \cong (X^{B\mathbb{Z}})^{B\mathbb{Z}} = LX^{B\mathbb{Z}}$$

where the first map is restriction along the multiplication $B\mathbb{Z} \times B\mathbb{Z} \to B\mathbb{Z}$ and the isomorphism is the associativity isomorphism of the sSet-module structure of dSt_k. The commutativity of the first diagram of (29) then follows from the associativity of the multiplication on $B\mathbb{Z}$. Composing with the monoid map $\mathscr{O}\colon \mathrm{Map}(LX, LX) \to \mathrm{Map}(\mathscr{O}(LX), \mathscr{O}(LX))$, we obtain a group action of $B\mathbb{Z}$ on the simplicial k-algebra $\mathscr{O}(LX)$.

Let us denote by $U\colon \mathsf{sAlg}_k \to \mathsf{sMod}_k$ the forgetful functor. If K is a simplicial set, U does not commute with $K \otimes ?$, but it does commute with its right adjoint $?^K$. This is a trivial consequence of the fact that U has a left adjoint which commutes with colimits and hence with $K\otimes ?$. (The same argument applied to the forgetful functor to sSet shows that in both categories A^K is $\mathrm{Map}_{\mathsf{sSet}}(K, A)$ with the algebraic structure coming from the target.) It follows that U has a structure of simplicial functor given by

$$\mathrm{Map}(A, B)_n = \mathsf{sComm}_k(A, B^{\Delta^n}) \to \mathsf{sMod}_k(UA, UB^{\Delta^n}) = \mathrm{Map}(UA, UB)_n.$$

Therefore we also obtain a group action of $B\mathbb{Z}$ on the simplicial k-module $U(\mathscr{O}(LX))$. This action is readily made explicit. For $x \in \mathbb{Z}^n$ a generator of $k[B\mathbb{Z}]_n$ and $\otimes_{y\in\mathbb{Z}^n} a_y$ is an element of $\mathscr{O}(LX)_n = (B\mathbb{Z} \otimes \mathscr{O}(X))_n$, then

$$x(\otimes_{y\in\mathbb{Z}^n} a_y) = \otimes_{y\in\mathbb{Z}^n} a_{y-x}.$$

The main observation is that this structure corresponds to the cyclic structure of the Hochschild complex.

Theorem 48. *Let $X \in \mathsf{dSt}_k$ and let $n \geq 0$. Then there is a natural isomorphism of k-modules*

$$HC_n^-(\mathbf{L}\mathscr{O}(X)) \cong \pi_n(\mathbf{R}\,\mathrm{fix}_{B\mathbb{Z}}\,\mathbf{L}\mathscr{O}\mathbf{R}L(X)).$$

It fits into a commutative square

$$\begin{array}{ccc} HC_n^-(\mathbf{L}\mathscr{O}(?)) & \longrightarrow & HH_n(\mathbf{L}\mathscr{O}(?)) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \pi_n(\mathbf{R}\,\mathrm{fix}_{B\mathbb{Z}}\,\mathbf{L}\mathscr{O}\mathbf{R}L(?)) & \to & \pi_n(\mathbf{L}\mathscr{O}\mathbf{R}L(?)) \end{array}$$

of functors $\mathrm{Ho}\,\mathsf{dSt}_k \to \mathsf{Mod}_k$ in which: the top arrow is the canonical map of §1.1; the right arrow is the isomorphism of Theorem 47; and the bottom arrow is induced by inclusion of fixed points.

4.3 The Chern character revisited

In this section we shall explain how to define the Chern character of a vector bundle on a derived stack, as sketched in [**TV08**, §3], and we shall prove that this new construction coincides through the identifications of the previous section with that of Chapter 1 when applied to a commutative k-algebra that is cofibrant as a constant simplicial commutative k-algebra (or more generally whose underlying k-module is flat). We shall only make the construction precise for affine derived stacks, but the same construction will define the Chern character of a vector bundle on an arbitrary derived stack X once we know the existence of a monoidal model category of vector bundles Vect_X that

- the canonical map $N(\mathsf{Vect}_X^{cw}) \to \mathbf{R}\,\mathrm{Map}(X, \mathrm{Vect})$ is an isomorphism in $\mathrm{Ho}\,\mathsf{sSet}$;
- every object of $\mathrm{Ho}\,\mathsf{Vect}_X$ is dualizable.

The idea of the construction is the following. Start with a vector bundle V on a derived stack X, and pull back V through the evaluation at zero $p\colon \mathbf{R}L(X) \to X$. Then the vector bundle $p^*(V)$ comes equipped with a canonical "monodromy" automorphism, and its trace is by definition the Chern character of V.

Let $X \in \mathsf{dSt}_k$. The natural adjunction map

$$B\mathbb{Z} \otimes^{\mathbf{L}} \mathbf{R}L(X) \to X$$

induces

$$\mathbf{R}_{\text{ét}} \operatorname{Map}(X, \mathrm{Vect}) \to \mathbf{R}_{\text{ét}} \operatorname{Map}(B\mathbb{Z} \otimes^{\mathbf{L}} \mathbf{R}L(X), \mathrm{Vect}) \cong \mathbf{R} \operatorname{Map}(B\mathbb{Z}, \mathbf{R}_{\text{ét}} \operatorname{Map}(\mathbf{R}L(X), \mathrm{Vect})).$$

Taking connected components we obtain a map of sets

$$[X, \mathrm{Vect}] \to [B\mathbb{Z}, \mathbf{R}_{\text{ét}} \operatorname{Map}(\mathbf{R}L(X), \mathrm{Vect})] \tag{31}$$

natural for $X \in \operatorname{Ho} \mathsf{dSt}_k$ (the square brackets on the left are the hom sets of $\operatorname{Ho}\mathsf{dSt}_k$). From now on we assume that $X = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$ is an affine derived stack. Let $\tilde{A} = B\mathbb{Z} \otimes^{\mathbf{L}} A$ be the topological Hochschild homology of A. Although it is not necessary, we will assume throughout that A is cofibrant so that we need not worry about taking cofibrant replacements to compute $\tilde{A}$. Thus $\tilde{A} = B\mathbb{Z} \otimes A$. By the derived Yoneda lemma, $\mathbf{R}_{\text{ét}} \operatorname{Map}(X, \mathrm{Vect})$ is naturally equivalent to $(R_{\text{ét}}\mathrm{Vect})(A)$. Recall that Vect is a derived stack and hence that $R_{\text{ét}}\mathrm{Vect}$ is just a pointwise fibrant replacement; we simply write $R\mathrm{Vect}(A)$ for $(R_{\text{ét}}\mathrm{Vect})(A)$. In particular, $\pi_n(R\mathrm{Vect}(A)) = \pi_n(\mathrm{Vect}(A))$. Therefore (31) becomes a natural map

$$\pi_0(\mathrm{Vect}(A)) \to [B\mathbb{Z}, R\mathrm{Vect}(\tilde{A})].$$

An element on the left is an isomorphism class in $\operatorname{Ho} \mathsf{sMod}_A$ of some perfect A-module M. Its image on the right is a homotopy class of maps of simplicial sets $B\mathbb{Z} \to R\mathrm{Vect}(\tilde{A})$. In particular it induces a well-defined map of sets

$$\pi_0(B\mathbb{Z}) = \{*\} \to \pi_0(\mathrm{Vect}(\tilde{A})), \tag{32}$$

pointing to the isomorphism class in $\operatorname{Ho} \mathsf{sMod}_{\tilde{A}}$ of some perfect $\tilde{A}$-module $\tilde{M}$, and a well-defined map of groups

$$\pi_1(B\mathbb{Z}, *) \cong \mathbb{Z} \to \pi_1(\mathrm{Vect}(\tilde{A}), \tilde{M}). \tag{33}$$

We are going to identify (32) and (33).

The inclusion $* \to B\mathbb{Z}$ induces for any X a map $\mathbf{R}L(X) \to B\mathbb{Z} \otimes^{\mathbf{L}} \mathbf{R}L(X)$ in $\operatorname{Ho}\mathsf{dSt}_k$, and therefore a map $\mathbf{R}L(X) \to X$. For $X = \mathbf{R}\underline{h}_{\operatorname{Spec} A}$ this is the map

$$e\colon A \to B\mathbb{Z} \otimes A \tag{34}$$

which in level n is the inclusion of A_n into the tensor component of $\bigotimes_{y \in \mathbb{Z}^n} A_n$ indexed by 0.

Lemma 49. *$\tilde{M}$ is obtained from M by extension of scalars along (34).*

Proof. Consider the diagram

$$\begin{array}{ccccc}
\pi_0(\mathrm{Vect}(A)) & \to & [B\mathbb{Z} \otimes^{\mathbf{L}} \mathbf{R}L(X), \mathrm{Vect}] & \xrightarrow{\cong} & [B\mathbb{Z}, R\mathrm{Vect}(\tilde{A})] \\
& \searrow^{e_*} & \downarrow & & \downarrow \\
& & \pi_0(\mathrm{Vect}(\tilde{A})) & \xrightarrow{\cong} & [*, R\mathrm{Vect}(\tilde{A})].
\end{array}$$

Here $e_* = \pi_0(\mathrm{Vect}(e))$ and the vertical maps are induced by $* \to B\mathbb{Z}$. The triangle is obtained from a commutative triangle by applying the functor $[?, \mathrm{Vect}]$ and so it is commutative. The square is also commutative by naturality of the horizontal adjunction isomorphisms. Travelling along the bottom path, the connected component of M goes to the homotopy class of maps $* \to R\mathrm{Vect}(\tilde{A})$ pointing to the connected component of $e_*(M)$. The commutativity of the diagram says that the homotopy class of $* \to R\mathrm{Vect}(\tilde{A})$ factors through the homotopy class of $B\mathbb{Z} \to R\mathrm{Vect}(\tilde{A})$ and so the π_0 of the latter also points to the connected component of $e_*(M)$, i.e., $\tilde{M} \cong e_*(M)$. ∎

Thus $\tilde{M} = M \otimes_A (B\mathbb{Z} \otimes A)$. This tensor product is formed by replacing in $(B\mathbb{Z} \otimes A)_n$ the factor A_n indexed by 0 by M_n:

$$\tilde{M}_n = M_n \otimes_k \bigotimes_{y \in \mathbb{Z}^n - \{0\}} A_n.$$

The identification of (33) requires more work. We begin with a description of the first homotopy group of $\mathrm{Vect}(A)$ (that applies to the nerve of any category). It is proved in [**DK80**, 5.5 (ii)] that $\Omega_M \mathrm{Vect}(A)$ is equivalent to the subsimplicial set $\mathrm{Aut}(M)$ of $\mathbf{R}\,\mathrm{Map}(M, M)$ spanned by the connected components corresponding to isomorphisms in $\mathrm{Ho}\,\mathsf{sMod}_A$. In particular, we have

$$\pi_1(\mathrm{Vect}(A), M) = \pi_0(\Omega_M \mathrm{Vect}(A)) \cong \pi_0(\mathrm{Aut}(M)) = \mathrm{Aut}_{\mathrm{Ho}\,\mathsf{sMod}_A}(M).$$

This bijection is induced by sending a self-equivalence of M to its image in the homotopy category. Thus, (33) becomes a map of groups

$$\mathbb{Z} \to \pi_1(\mathrm{Vect}(\tilde{A}), \tilde{M}) \cong \mathrm{Aut}_{\mathrm{Ho}\,\mathsf{sMod}_{\tilde{A}}}(\tilde{M}), \tag{35}$$

i.e., an action of $\mathbb{Z}$ on $\tilde{M}$ in the homotopy category. Consider now the functor $\mathrm{Vect}' \colon \mathsf{sComm}_k \to \mathsf{sSet}$ defined as Vect except that we apply the groupoid completion functor to Vect_A^{cw} before taking the nerve. There is a canonical natural transformation $\gamma \colon \mathrm{Vect} \to \mathrm{Vect}'$. The same result from [**DK80**] tells us that $\Omega_M \mathrm{Vect}'(A)$ is equivalent to the constant simplicial set $\mathrm{Aut}_{\mathrm{Ho}\,\mathsf{sMod}_A}(M)$, and it follows that the map $\mathrm{Vect}(A) \to \mathrm{Vect}'(A)$ induces an isomorphism on π_1 for every choice of base point (and it is of course the identity in degree 0). Being the nerve of a groupoid, $\mathrm{Vect}'(A)$ is a fibrant simplicial set. Since $R_{\text{ét}}\mathrm{Vect}$ is just a pointwise fibrant replacement of Vect, we can choose $R_{\text{ét}}\mathrm{Vect}$ by applying a functorial factorization in sSet to the natural transformation γ, i.e., γ factors as

$$\mathrm{Vect} \to R_{\text{ét}}\mathrm{Vect} \to \mathrm{Vect}'$$

where the first map is a pointwise trivial cofibration and the second map is a pointwise fibration.

Let us now fix a cofibrant and perfect simplicial A-module M representing an element of $\pi_0(\mathrm{Vect}(A))$ (always assuming that A itself is cofibrant). It corresponds by the simplicial Yoneda lemma to a map $f \colon h_{\mathrm{Spec}\,A} \to \mathrm{Vect}$. We will prove below that f has a lift $g \colon \underline{h}_{\mathrm{Spec}\,A} \to \mathrm{Vect}'$ as in the diagram

$$\begin{array}{ccc} h_{\mathrm{Spec}\,A} & \xrightarrow{f} & \mathrm{Vect} \\ \downarrow & & \downarrow{\scriptstyle\gamma} \\ \underline{h}_{\mathrm{Spec}\,A} & \overset{g}{\dashrightarrow} & \mathrm{Vect}'. \end{array}$$

Once g has been defined we can work explicitly at the level of representatives: a representative of the homotopy class of maps $B\mathbb{Z} \to R\mathrm{Vect}(\tilde{A}) \to \mathrm{Vect}'(\tilde{A})$ corresponding to M will be the image of g through the composition

$$\mathsf{dSt}_k(\underline{h}_{\mathrm{Spec}\,A}, \mathrm{Vect}') \to \mathsf{dSt}_k(B\mathbb{Z} \otimes h_{\mathrm{Spec}\,\tilde{A}}, \mathrm{Vect}') \cong \mathsf{sSet}(B\mathbb{Z}, \mathrm{Map}(h_{\mathrm{Spec}\,\tilde{A}}, \mathrm{Vect}')),$$

where the first map is precomposition by

$$B\mathbb{Z} \otimes h_{\mathrm{Spec}\,\tilde{A}} \to B\mathbb{Z} \otimes \underline{h}_{\mathrm{Spec}\,\tilde{A}} \cong B\mathbb{Z} \otimes \underline{h}^{B\mathbb{Z}}_{\mathrm{Spec}\,A} \to \underline{h}_{\mathrm{Spec}\,A}.$$

A computation yields the following formula. An element $z \in \mathbb{Z}$ will be sent to the equivalence class in $\pi_1(\mathrm{Vect}'(\tilde{A}), \tilde{M}) \cong \pi_1(\mathrm{Vect}(\tilde{A}), \tilde{M})$ of the 1-simplex

$$g_{\tilde{A},1}(\omega(z)) \in \mathrm{Vect}'(\tilde{A})_1 \tag{36}$$

where $\omega(z) \colon \Delta^1 \otimes A \to \tilde{A}$ is the morphism in sComm_k induced by the classifying map $\Delta^1 \to B\mathbb{Z}$ of z. Note that $\omega(z)$ is a homotopy from (34) to itself.

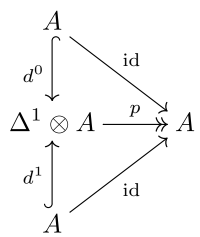
Lemma 50. *Let K be a simplicial set and C a category. Then the map*

$$\mathsf{sSet}(K, N\mathsf{C}) \to \mathsf{s}_1\mathsf{Set}(i_1^*(K), i_1^*(N\mathsf{C}))$$

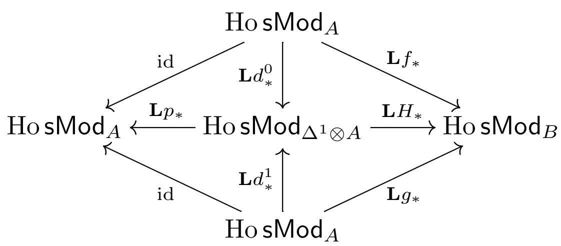
is injective and its image consists of those morphisms $g\colon i_1^*(K) \to i_1^*(N\mathsf{C})$ satisfying $g_1(d_1x) = g_1(d_0x) \circ g_1(d_2x)$ for all $x \in K_2$.

Proof. The map $\mathsf{sSet}(K, N\mathsf{C}) \to \mathsf{s}_2\mathsf{Set}(i_2^*(K), i_2^*(N\mathsf{C}))$ is a bijection because nerves of categories are 2-coskeletons. Let $g\colon i_2^*(K) \to i_2^*(N\mathsf{C})$. Clearly g_2 is determined by g_1 since a 2-simplex in $N\mathsf{C}$ is determined by its two external faces. This proves injectivity. Explicitly one has $g_2(x) = (g_1(d_2x), g_1(d_0x))$. It is then easy to check that if g_0 and g_1 are given forming a map in $\mathsf{s}_1\mathsf{Set}$ and if g_2 is defined by this formula, then $g_2s_i = s_ig_1$ for $i = 0, 1$ and $g_1d_i = d_ig_2$ for $i = 0, 2$, so a necessary and sufficient condition for g_0 and g_1 to extend to a map in $\mathsf{s}_2\mathsf{Set}$ is $g_1d_1 = d_1g_2$. This is precisely the condition of the lemma. □

We now define the lift g. Since $h_{\operatorname{Spec} A}$ takes values in constant simplicial sets, for g to extend f it is necessary and sufficient that $g_0 = \gamma_0 f_0 = f_0$, so it remains to define g_1 such that g_0 and g_1 form a map in $\mathsf{s}_1\mathsf{Set}$ and to check the additional condition of Lemma 50. Let $B \in \mathsf{sComm}_k$ and $H \in \underline{h}_{\operatorname{Spec} A}(B)_1$, i.e., H is a homotopy $\Delta^1 \otimes A \to B$ between its two faces $f\colon A \to B$ and $g\colon A \to B$. We consider the commutative diagram in sComm_k



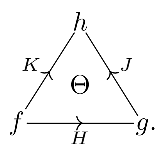
in which all maps are weak equivalences. It induces a diagram of functors



which is commutative up to natural isomorphism and in which $\mathbf{L}p_*$, $\mathbf{L}d^0_*$, and $\mathbf{L}d^1_*$ are equivalences of categories. It shows that the functors $\mathbf{L}f_*$, $\mathbf{L}H_*\mathbf{R}p^*$, and $\mathbf{L}g_*$ are all naturally isomorphic, explicit isomorphisms being given by

$$\begin{aligned}
\epsilon_0\colon \mathbf{L}f_* &= \mathbf{L}H_*\mathbf{L}d^0_* = \mathbf{L}H_*\mathbf{L}d^0_*\mathbf{R}(d^0)^*\mathbf{R}p^* \to \mathbf{L}H_*\mathbf{R}p^*,\\
\epsilon_1\colon \mathbf{L}g_* &= \mathbf{L}H_*\mathbf{L}d^1_* = \mathbf{L}H_*\mathbf{L}d^1_*\mathbf{R}(d^1)^*\mathbf{R}p^* \to \mathbf{L}H_*\mathbf{R}p^*,
\end{aligned}$$

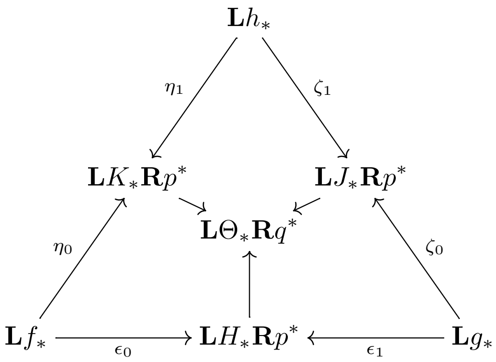
where the nonidentity maps are induced by the counits of the equivalences $(\mathbf{L}d^0_*, \mathbf{R}(d^0)^*)$ and $(\mathbf{L}d^1_*, \mathbf{R}(d^1)^*)$. We define $g_1(H)$ to be $\epsilon_{1,M}^{-1}\epsilon_{0,M}\colon M \otimes_f B \to M \otimes_g B$. Then g_0 and g_1 are compatible with d_0, d_1, and s_0 by construction. It remains to prove that for $\Theta\colon \Delta^2 \otimes A \to B$ a 2-simplex of $\underline{h}_{\operatorname{Spec} A}(B)$, we have $g_1(K) = g_1(J) \circ g_1(H)$, where



Let ϵ_0 and ϵ_1 be the natural isomorphisms induced by H as above, and let ζ_0 and ζ_1 (resp. η_0 and η_1) be the ones induced by J (resp. by K). Then we must check that

$$\zeta_{1,M}^{-1}\zeta_{0,M}\epsilon_{1,M}^{-1}\epsilon_{0,M} = \eta_{1,M}^{-1}\eta_{0,M}.$$

The proof proceeds by observing that each of these natural isomorphisms factors through $\mathbf{L}\Theta_*\mathbf{R}q^*$ where $q\colon \Delta^2 \otimes A \twoheadrightarrow A$ is the projection: there is a diagram of natural isomorphisms



in which all three squares are commutative.

Applying the formula (36), we find that the automorphism of $\tilde{M}$ image of $z \in \mathbb{Z}$ by (35) is the composition

$$\tilde{M} = M \otimes^{\mathbf{L}}_{\omega(z)d^0} \tilde{A} \to \mathbf{R}p^*(M) \otimes^{\mathbf{L}}_{\omega(z)} \tilde{A} \leftarrow M \otimes^{\mathbf{L}}_{\omega(z)d^1} \tilde{A} = \tilde{M}. \tag{37}$$

Everything we have done so far would work as well with Qcoh in place of Vect, and only now do we use that M is perfect. Recall that perfect simplicial A-modules are the dualizable objects in the closed symmetric monoidal category $\mathrm{Ho}\,\mathsf{sMod}_A$. Therefore we can consider for any $M \in \mathsf{sMod}_A$ the trace map

$$\mathrm{tr}\colon [M, M] \to [A, A]$$

which is the composition

$$[M, M] \cong [A \otimes^{\mathbf{L}}_A M, M] \cong [A, \mathbf{R}\,\mathrm{Hom}(M, M)] \cong [A, M \otimes^{\mathbf{L}}_A \mathbf{R}\,\mathrm{Hom}_A(M, A)] \to [A, A].$$

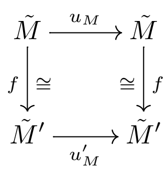
Let $u_M\colon \tilde{M} \to \tilde{M}$ denote the image of $1 \in \mathbb{Z}$ by (35). The *Chern character* of M is $\mathrm{tr}(u_M) \in [A, A]$. We must prove that it does not depend on the choice of the representative $\tilde{M}$. If $f\colon \tilde{M} \to \tilde{M}'$ is an equivalence between cofibrant objects in $\mathsf{sMod}_{\tilde{A}}$, i.e. a 1-simplex of $\mathrm{Vect}(\tilde{A})$, and $u'_M\colon \tilde{M}' \to \tilde{M}'$ is the image of $1 \in \mathbb{Z}$ by (35), the isomorphism

$$\pi_1(\mathrm{Vect}(\tilde{A}), \tilde{M}) \to \pi_1(\mathrm{Vect}(\tilde{A}), \tilde{M}')$$

induced by f corresponds to the isomorphism

$$f \circ ? \circ f^{-1}\colon\ \mathrm{Aut}_{\mathrm{Ho}\,\mathsf{sMod}_{\tilde{A}}}(\tilde{M}) \to \mathrm{Aut}_{\mathrm{Ho}\,\mathsf{sMod}_{\tilde{A}}}(\tilde{M}')$$

and so there is a commutative diagram



which implies that $\mathrm{tr}(u_M) = \mathrm{tr}(u'_M)$.

Recall that for $A \in \mathsf{sComm}_k$ there is a Quillen adjunction

$$\pi_0\colon \mathsf{sMod}_A \rightleftarrows \mathsf{Mod}_{\pi_0(A)} :i.$$

By definition of perfect simplicial modules, M perfect implies $\pi_0(M)$ finitely generated and projective. In other words, the total derived functor $\mathbf{L}\pi_0$ preserves dualizable objects.

Proposition 51. *Let $A \in \mathsf{sComm}_k$ and let $M \in \mathsf{sMod}_A$ be perfect. Then the square*

$$\begin{array}{ccc} [M,M] & \xrightarrow{\mathrm{tr}} & [A,A] \\ {\scriptstyle \mathbf{L}\pi_0}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \mathbf{L}\pi_0} \\ [\pi_0(M),\pi_0(M)] & \xrightarrow[\mathrm{tr}]{} & [\pi_0(A),\pi_0(A)] \end{array}$$

is commutative.

Proof. Recall that the (nonunital) monoidal structure gives us a canonical map

$$\pi_0(M \otimes^{\mathbf{L}}_A N) \to \pi_0(M) \otimes_{\pi_0(A)} \pi_0(N)$$

that coincides with the edge morphism of the Künneth spectral sequence. It follows from Lemma 17 that this map is an isomorphism if M is perfect. By adjunction, we obtain a canonical map

$$\pi_0(\mathbf{R}\operatorname{Hom}_A(M,N)) \to \operatorname{Hom}_{\pi_0(A)}(\pi_0(M),\pi_0(N)),$$

which on the underlying sets is the bijection of Lemma 32. We consider the diagram

$$\begin{array}{ccccccc}
[M,M] & \xrightarrow{\cong} & [A,\mathbf{R}\operatorname{Hom}(M,M)] & \xleftarrow{\cong} & [A, M\otimes^{\mathbf{L}} M^\vee] & \longrightarrow & [A,A] \\
\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\
 & & [\pi_0(A),\pi_0(\mathbf{R}\operatorname{Hom}(M,M))] & \xleftarrow{\cong} & [\pi_0(A),\pi_0(M\otimes^{\mathbf{L}} M^\vee)] & \longrightarrow & [\pi_0(A),\pi_0(A)] \\
 & & {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow & & \big\Vert \\
 & & & & [\pi_0(A),\pi_0(M)\otimes\pi_0(M^\vee)] & & \\
 & & & & {\scriptstyle\cong}\big\downarrow & & \\
[\pi_0(M),\pi_0(M)] & \xrightarrow{\cong} & [\pi_0(A),\operatorname{Hom}(\pi_0(M),\pi_0(M))] & \xleftarrow{\cong} & [\pi_0(A),\pi_0(M)\otimes\pi_0(M)^\vee] & \to & [\pi_0(A),\pi_0(A)]
\end{array}$$

in which the top and bottom rows are the trace maps. The commutativity of the two small rectangles is clear by functoriality of π_0. We check the commutativity of the other three rectangles on elements, and we can assume that M is cofibrant. Let $[f] \in [M,M]$; its image in $[A,\operatorname{Hom}(M,M)]$ has a representative sending $1 \in A_0$ to f, so its image in $[\pi_0(A),\operatorname{Hom}(\pi_0(M),\pi_0(M))]$ sends $[1]$ to $\pi_0(f)$. This is obviously what the other image of $[f]$ does as well. The last two rectangles come from the diagram

$$\begin{array}{ccccc}
\pi_0(\operatorname{Hom}(M,M)) & \xleftarrow{\cong} & \pi_0(M\otimes M^\vee) & \longrightarrow & \pi_0(A) \\
\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow & & \big\Vert \\
 & & \pi_0(M)\otimes\pi_0(M^\vee) & & \\
 & & {\scriptstyle\cong}\big\downarrow & & \\
\operatorname{Hom}(\pi_0(M),\pi_0(M)) & \xleftarrow{\cong} & \pi_0(M)\otimes\pi_0(M)^\vee & \longrightarrow & \pi_0(A)
\end{array}$$

by applying the functor $[\pi_0(A), ?]$. Let $[x\otimes\xi] \in \pi_0(M\otimes M^\vee)$. Both images of $[x\otimes\xi]$ in $\operatorname{Hom}(\pi_0(M),\pi_0(M))$ are then $\pi_0(f)$ where $f_n(y) = \xi_n(y)s(x) \in M_n$ ($s(x)$ is the degeneracy of x in degree n). The commutativity of the second rectangle is equally clear. □

Theorem 52. *Let A be a flat commutative k-algebra and let M be an f.g.p. A-module. Then the Chern character of M in $HH_0(A) = \pi_0(\tilde{A}) = A$ as defined in §1.3 coincides with the Chern character of M defined above.*

Proof. Both maps in (37) agree on π_0. Therefore the automorphism $u_M : \tilde{M} \to \tilde{M}$ is such that $\pi_0(u_M) = \mathrm{id}$. Since $\pi_0(\tilde{M}) = M$, Proposition 51 implies that $\mathrm{tr}(u_M)$ is the trace of the identity on M. This is the same thing as the trace of an idempotent representing M. □

The next step would be to prove that the Chern character is "S^1-equivariant", i.e., that it lifts to the negative cyclic homology $HC_0^-(A)$. There could be many such lifts, so we would also need a way to select a natural one. This is expected to be a nontrivial result. We have already solved it for constant simplicial k-algebras in Chapter 1 using Morita naturality, but we do not know if this is true for arbitrary simplicial commutative k-algebra, let alone for arbitrary derived stacks. It is possible however that the proof of Chapter 1 can be adapted to simplicial commutative k-algebras using a homotopical generalization of the category Mor_k (defined in the next chapter) and derived Morita theory.

5 Categorical sheaves

5.1 The homotopy theory of simplicial categories

Fix a commutative ring k. A *simplicial k-category* is a category enriched in the closed symmetric monoidal category sMod_k of simplicial k-modules. A morphism of simplicial k-categories is an sMod_k-enriched functor. Let sCat_k denote the category of (small) simplicial k-categories. Since the monoidal category sMod_k is closed, it is itself a simplicial k-category (although it is not an object of sCat_k). For a simplicial k-category C, the category of sMod_k-enriched functor from C to sMod_k with sMod_k-enriched natural transformations as morphisms is called the category of *simplicial C-modules* and is denoted by sMod_C; it is in fact an sMod_k-module with tensor and cotensor defined objectwise. Observe that a simplicial k-category A with a single object is nothing else than a simplicial k-algebra, and a simplicial A-module as just defined is the same thing as a left simplicial A-module.

Let Cat_k denote the category of small Mod_k-enriched categories (also called *k-categories*). There is a functor $\pi_0 \colon \mathsf{sCat}_k \to \mathsf{Cat}_k$ that sends a simplicial k-category C to the k-category $\pi_0(C)$ with the same objects as C and with $\pi_0(C)(x,y) = \pi_0(C(x,y))$. Composition in $\pi_0(C)$ is defined using the canonical map of k-modules $\pi_0(C(x,y))\otimes_k\pi_0(C(y,z)) \to \pi_0(C(x,y)\otimes_k C(y,z))$. If f is a morphism of simplicial k-categories, $\pi_0(f)$ is such that $\pi_0(f)_{x,y} = \pi_0(f_{x,y})$. It is clear that π_0 is left adjoint to the obvious fully faithful functor $\mathsf{Cat}_k \to \mathsf{sCat}_k$. This functor also has a right adjoint $C \mapsto C_0$ defined by $C_0(x,y) = C(x,y)_0$, and there is a natural transformation $?_0 \to \pi_0$. A morphism of C_0 is a *homotopy equivalence* if its image in $\pi_0(C)$ is invertible. We shall occasionnaly view $\pi_0(C)$ or C_0 as mere categories using the forgetful functor $\mathsf{Cat}_k \to \mathsf{Cat}$, and we observe that this forgetful functor reflects equivalences (because the forgetful functor $\mathsf{Mod}_k \to \mathsf{Set}$ reflects isomorphisms).

Theorem 53. *1. There is a cofibrantly generated model structure on sCat_k in which a map $f \colon C \to D$ of simplicial k-categories is*

- *an equivalence if for every objects x and y in C, $f_{x,y}$ is an equivalence of simplicial k-modules, and if $\pi_0(f)$ is an equivalence of k-categories;*
- *a fibration if for every objects x and y of C, $f_{x,y}$ is a fibration and if for every $x \in C$, every $y' \in D$, and every homotopy equivalence $u \in D_0(f(x), y')$, there exists $y \in C$ and a homotopy equivalence $u \in C_0(x,y)$ such that $f_0(u) = v$.*

2. Let C be a simplicial k-category. There is a cofibrantly generated model structure on sMod_C in which a map $f \colon M \to N$ of simplicial C-modules is an equivalence (resp. a fibration) if it is a pointwise equivalence (resp. a pointwise fibration) in sMod_k. With this structure, sMod_C is an sMod_k-model category.

Proof. For 1, see [**Ber04**] or [**Tab07a**]. Since sMod_k is cofibrantly generated, 2 follows from an sMod_k-enriched version of [**Hir03**, 11.6.1]. □

There is a duality automorphism $C \mapsto C^{\mathrm{op}}$ of sCat_k defined by $\mathrm{Ob}(C^{\mathrm{op}}) = \mathrm{Ob}\, C$, $C^{\mathrm{op}}(x,y) = C(y,x)$ and $(f^{\mathrm{op}})_{x,y} = f_{y,x}$. It clearly preserves the model structure of the theorem.

Conjecture 54. *There exists a cofibrant replacement functor Q on sCat_k such that $QC \to C$ is the identity on objects and such that $Q(C^{\mathrm{op}}) = (QC)^{\mathrm{op}}$.*

Conjecture 55. *If C is a cofibrant simplicial k-category, then*

1. *for every objects x and y, $C(x,y)$ is cofibrant in sMod_k;*
2. *any cofibration in sMod_C is a pointwise cofibration.*

There are several other model structures of interest on sCat_k with larger sets of equivalences. For instance, there is a left Bousfield localization of the above structure in which the equivalences are the *Morita equivalences*, i.e., the simplicial k-functors $C \to D$ inducing by precomposition (see below) an equivalence of categories $\mathrm{Ho}\,\mathsf{sMod}_D \to \mathrm{Ho}\,\mathsf{sMod}_C$. To any simplicial k-category one can associate a mixed complex from which one can define various homology theories, and these turn out to be Morita invariant in the obvious sense. An interesting fact is that for any simplicial commutative k-algebra A there is a Morita equivalence between A and the simplicial k-category of perfect simplicial A-modules, and hence that one can define the Hochschild homology and cyclic homologies of A using this simplicial category.

Let $f\colon C \to D$ be a simplicial k-functor. It induces an adjunction

$$f_*\colon \mathsf{sMod}_C \rightleftarrows \mathsf{sMod}_D : f^* \tag{38}$$

where $f^*(M) = Mf$ and f_* exists by the theory of enriched Kan extensions. Since f^* clearly preserves equivalences and fibrations, this is a Quillen adjunction.

Conjecture 56. *If $f\colon C \to D$ is an equivalence in sCat_k, then (38) is a Quillen equivalence.*

The *tensor product* of two simplicial k-categories C and D is defined by

- $\mathrm{Ob}(C \otimes D) = \mathrm{Ob}\, C \times \mathrm{Ob}\, D$ and
- $(C \otimes D)((x, y), (x', y')) = C(x, x') \otimes_k D(y, y')$

with the obvious compositions. This clearly defines a symmetric monoidal structure on sCat_k whose unit is the simplicial k-algebra k. Note that $(C \otimes D)^{\mathrm{op}} = C^{\mathrm{op}} \otimes D^{\mathrm{op}}$.

There is also a tensor product of bimodules defined as follows. Let C, D, and E be simplicial k-categories. Let M be a simplicial $C \otimes D^{\mathrm{op}}$-module and N a simplicial $D \otimes E^{\mathrm{op}}$-module. One defines a simplicial $C \otimes E^{\mathrm{op}}$-module $M \otimes_D N$ by the formula

$$(M \otimes_D N)(x, z) = M(x, ?) \otimes_D N(?, z)$$

where the right-hand side is an sMod_k-enriched coend. Any simplicial k-category C can be seen as a simplicial $C \otimes C^{\mathrm{op}}$-module defined by its sMod_k-enriched hom's.

Conjecture 57. 1. *Let Q be a cofibrant replacement functor on sCat_k. Then the functor*

$$\mathsf{sCat}_k \times \mathsf{sCat}_k \to \mathsf{sCat}_k, \quad (C, D) \mapsto QC \otimes D$$

preserves equivalences.

2. *Let C, D, and E be cofibrant simplicial k-categories. The tensor product of simplicial bimodules $\otimes_D\colon \mathsf{sMod}_{C\otimes D^{\mathrm{op}}} \times \mathsf{sMod}_{D\otimes E^{\mathrm{op}}} \to \mathsf{sMod}_{C\otimes E^{\mathrm{op}}}$ has a total left derived functor.*

A consequence of 1 is that the functor $\otimes\colon \mathsf{sCat}_k \times \mathsf{sCat}_k \to \mathsf{sCat}_k$ has a total left derived functor induced by any of the equivalence-preserving functors $(C, D) \mapsto QC \otimes D$, $(C, D) \mapsto C \otimes QD$, and $(C, D) \mapsto QC \otimes QD$.

Let C be a simplicial k-category. A simplicial C-module M is called *perfect* if it is homotopically finitely presented in the model category sMod_C, i.e., if for any filtered index category I and any functor $d\colon \mathsf{I} \to \mathsf{sMod}_C$, the canonical map

$$\mathbf{R}\,\mathrm{Map}(M, \underrightarrow{\mathrm{holim}}\, d) \to \underrightarrow{\mathrm{holim}}\, \mathbf{R}\,\mathrm{Map}(M, d)$$

is an isomorphism in $\mathrm{Ho}\,\mathsf{sSet}$. A $C \otimes D^{\mathrm{op}}$-module M is called right perfect if for every $x \in C$ the D^{op}-module $M(x, ?)$ is perfect. This notion of perfect simplicial module extends the one encountered in §3.4.

Conjecture 58. *Let C be the simplicial k-module k viewed as a simplicial k-category. Then a simplicial C-module is perfect if and only if it is perfect as a simplicial k-module.*

Using bimodules on can enhance the category sCat_k into a category $\mathsf{sCat}_k^{\mathrm{c}}$. The definition of $\mathsf{sCat}_k^{\mathrm{c}}$ is completely analogous to that of the category Mor_k studied in Chapter 1. Its objects are small simplicial k-categories, and the set of morphisms from C to D is the set of isomorphism classes of right perfect $C \otimes^{\mathbf{L}} D^{\mathrm{op}}$-modules in $\mathrm{Ho}\,\mathsf{sMod}_{C\otimes^{\mathbf{L}} D^{\mathrm{op}}}$. For definiteness we fix a cofibrant replacement functor Q on sCat_k satisfying Conjecture 54, and $C \otimes^{\mathbf{L}} D^{\mathrm{op}}$ is short for $QC \otimes QD^{\mathrm{op}}$. Composition is given by the derived tensor product of bimodules

$$\otimes^{\mathbf{L}}_{QD} \colon \operatorname{Ho}\mathsf{sMod}_{C\otimes^{\mathbf{L}}D^{\mathrm{op}}} \times \operatorname{Ho}\mathsf{sMod}_{D\otimes^{\mathbf{L}}E^{\mathrm{op}}} \to \operatorname{Ho}\mathsf{sMod}_{C\otimes^{\mathbf{L}}E^{\mathrm{op}}}.$$

There is a functor $I \colon \mathsf{sCat}_k \to \mathsf{sCat}^{\mathrm{c}}_k$ that is the identity on objects and sends a morphism $f \colon C \to D$ to the isomorphism class of the $C \otimes^{\mathbf{L}} D^{\mathrm{op}}$-module $I(f)$ defined by $(I(f))(x,y) = (QD)(y,(Qf)(x))$. It is "fully faithful" in the following sense: if $I(f) = I(g)$, the $\operatorname{Ho}\mathsf{sMod}_k$-enriched Yoneda lemma implies that f and g are naturally isomorphic as $\operatorname{Ho}\mathsf{sMod}_k$-enriched functors.†

The category $\mathsf{sCat}^{\mathrm{c}}_k$ has a symmetric monoidal structure $\otimes^{\mathbf{L}}$ defined on objects by $C \otimes^{\mathbf{L}} D = QC \otimes QD$. On morphisms it is induced by the functor

$$\operatorname{Ho}\mathsf{sMod}_{C\otimes^{\mathbf{L}}D^{\mathrm{op}}} \times \operatorname{Ho}\mathsf{sMod}_{E\otimes^{\mathbf{L}}F^{\mathrm{op}}} \to \operatorname{Ho}\mathsf{sMod}_{(C\otimes^{\mathbf{L}}E)\otimes^{\mathbf{L}}(D\otimes^{\mathbf{L}}F)^{\mathrm{op}}},$$
$$(M,N) \mapsto P, \quad P(x,z,y,w) = M(x,y) \otimes_k N(z,w).$$

It is straightforward to check that $I(f \otimes g) = I(f) \otimes I(g)$, so that I is a monoidal functor.

In the next section we shall use the following generalization of the notion of simplicial k-category. If A is a simplicial commutative k-algebra, the category sMod_A of simplicial modules over A is a closed symmetric monoidal category. We define an *simplicial A-category* to be a category enriched in sMod_A. All the definitions and results of this section extend to simplicial categories over simplicial commutative k-algebras. In particular, we can define the category $\mathsf{sCat}^{\mathrm{c}}_A$. A morphism $f \colon A \to B$ in sComm_k induces an obvious adjunction

$$f_* \colon \mathsf{sCat}_A \rightleftarrows \mathsf{sCat}_B : f^*$$

with $f_*(C)(x,y) = f_*(C(x,y))$ and $f^*(D)(x,y) = f^*(D(x,y))$, which is manifestly a Quillen adjunction. This adjunction extends to an adjunction

$$f_* \colon \mathsf{sCat}^{\mathrm{c}}_A \rightleftarrows \mathsf{sCat}^{\mathrm{c}}_B : f^*$$

in which f_* acts on bimodules by extending the scalars. More precisely, if C and D are two cofibrant simplicial A-categories and if M is a simplicial $C \otimes D^{\mathrm{op}}$-module, then

$$f_*(M) \colon f_*(C) \otimes_B f_*(D)^{\mathrm{op}} \to \mathsf{sMod}_B$$

is defined on objects by $f_*(M)(x,y) = f_*(M(x,y))$ and on the simplicial B-module of morphisms from (x,y) to (x',y') by

$$\begin{aligned} f_*C(x,x') \otimes_B f_*D(y',y) &\cong f_*(C(x,x') \otimes_A D(y',y)) \\ &\to f_*(\operatorname{Hom}_A(M(x,y), M(x',y'))) \to \operatorname{Hom}_B(f_*M(x,y), f_*M(x',y')). \end{aligned}$$

5.2 The Chern character of a categorical sheaf

Let $f \colon A \to B$ be a morphism of simplicial commutative k-algebras. It induces a functor

$$f_* \colon \mathsf{sCat}^{\mathrm{c}}_A \to \mathsf{sCat}^{\mathrm{c}}_B.$$

If $g \colon B \to C$ is another morphism in sComm_k, there is a canonical isomorphism $(gf)_* \cong g_* f_*$. Using the same trick as in §3.4 we can define for every $A \in \mathsf{sComm}_k$ a new category CQcoh_A varying *functorially* with A, together with an equivalence of categories $\mathsf{CQcoh}_A \to \mathsf{sCat}_A$ such that for every $f \colon A \to B$ the diagram

$$\begin{array}{ccc} \mathsf{CQcoh}_A & \xrightarrow{f_*} & \mathsf{CQcoh}_B \\ \downarrow & & \downarrow \\ \mathsf{sCat}^{\mathrm{c}}_A & \xrightarrow[f_*]{} & \mathsf{sCat}^{\mathrm{c}}_B \end{array}$$

†There is obviously some 2-categorical stuff going on here. In fact, $\mathsf{sCat}^{\mathrm{s}}_k$ can be viewed as a $(\infty,2)$-category since its categories of morphisms are connected components of simplicial categories, so the simplifications adopted here are quite extreme.

commutes up to a natural isomorphism. Composing with the nerve, we obtain a functor

$$\mathrm{CQcoh}\colon \mathsf{sComm}_k \to \mathsf{sSet}, \quad A \mapsto N(\mathsf{CQcoh}_A^{\cong})$$

where $\mathsf{CQcoh}_A^{\cong}$ is the subcategory of isomorphisms in CQcoh_A.

A simplicial A-category is called *saturated* if it is Morita equivalent to a cofibrant simplicial k-category C with a single object such that

- C is perfect as a simplicial k-module and
- C is perfect as a simplicial $C \otimes C^{\mathrm{op}}$-module.

Conjecture 59. *A simplicial A-category C is saturated if and only if it is dualizable in the monoidal category $\mathsf{sCat}_A^{\mathrm{c}}$.*

The full subcategory of $\mathsf{sCat}_A^{\mathrm{c}}$ consisting of saturated objects will be denoted by $\mathsf{sCat}_A^{\mathrm{s}}$.

Conjecture 60. *If $f\colon A \to B$ is a morphism in sComm_k, then $f_*\colon \mathsf{sCat}_A^{\mathrm{c}} \to \mathsf{sCat}_B^{\mathrm{c}}$ preserves saturated objects.*

Therefore CQcoh has a subfunctor

$$\mathrm{CVect}\colon \mathsf{sComm}_k \to \mathsf{sSet}, \quad A \mapsto N(\mathsf{CVect}_A^{\cong})$$

where CVect_A is the full subcategory of CQcoh_A consisting of objects whose image in $\mathsf{sCat}_A^{\mathrm{c}}$ is saturated.

Conjecture 61. CQcoh *and* CVect *are derived stacks.*

Assuming this one can proceed in exactly the same way as in §4.3. Given a simplicial k-category C over a cofibrant simplicial commutative k-algebra A, we obtain a well-defined map of sets

$$\pi_0(B\mathbb{Z}) = \{*\} \to \pi_0(\mathrm{CVect}(\tilde{A})),$$

where $\tilde{A} = B\mathbb{Z} \otimes A$, pointing to the saturated simplicial k-category $\tilde{C}$ obtained from C by extending the scalars along the inclusion $A \to \tilde{A}$, and a well-defined group action

$$\pi_1(B\mathbb{Z}, *) \cong \mathbb{Z} \to \pi_1(\mathrm{CVect}(\tilde{A}), \tilde{C}) \cong \mathrm{Aut}_{\mathsf{sCat}_{\tilde{A}}^{\mathrm{s}}}(\tilde{C}).$$

The image of $1 \in \mathbb{Z}$ is the isomorphism class of an invertible simplicial $\tilde{C} \otimes \tilde{C}^{\mathrm{op}}$-module. As a morphism in $\mathsf{sCat}_{\tilde{A}}^{\mathrm{s}}$, it has a trace which is an element of $\mathsf{sCat}_{\tilde{A}}^{\mathrm{s}}(\tilde{A}, \tilde{A})$, i.e., an isomorphism class in $\mathrm{Ho}\,\mathsf{sMod}_{\tilde{A}}$ of a perfect simplicial $\tilde{A}$-module. This perfect simplicial module on the topological Hochschild homology of A is by definition the Chern character of the simplicial k-category C.

Finally, it is expected that CQcoh and CVect classify a notion of categorical sheaf on derived stacks, so that the above construction extends to arbitrary derived stacks. The Chern character of a categorical sheaf on a derived stack X is thus a vector bundle on its derived loop space $\mathbf{R}L(X)$.

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