A TRIVIAL REMARK ON THE NISNEVICH TOPOLOGY

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Abstract. We observe that all existing definitions of the Nisnevich topology are equivalent.

Let $X$ be a quasi-compact quasi-separated scheme and let $Y \to X$ be an étale morphism which is surjective on $k$-points for every field $k$. Then there exists a sequence $\emptyset = Z_0 \subset Z_{n-1} \subset \cdots \subset Z_1 = X$ of finitely presented closed subschemes such that $Y \to X$ admits a section over $Z_{i-1} \setminus Z_i$ for all $i$. This generalizes [MV99, §3, Lemma 1.5], where $X$ is assumed noetherian.

Here is the proof. Consider the set $\Phi$ of all closed subschemes $Z \subset X$ for which the map $Y \times_X Z \to Z$ does not admit such a sequence. If $Z$ is a cofiltered intersection $\bigcap \alpha Z_\alpha$ and $Z \notin \Phi$, then there exists $\alpha$ such that $Z_\alpha \notin \Phi$, by [Gro66, Proposition 8.6.3 and Théorème 8.8.2 (i)]. In particular, $\Phi$ is inductively ordered. Next, we note that any étale morphism that splits over a maximal point $x$ splits over some open neighborhood of $x$: indeed, the local ring $\mathcal{O}_x$ is henselian, since its reduction is a field. If $Z \in \Phi$, then $Z$ is nonempty and hence has a maximal point. Thus, $Y \times_X Z \to Z$ splits over some nonempty open subscheme of $Z$, which may be chosen to have a finitely presented closed complement $W \subset Z$, by [TT90, Lemma 2.6.1 (c)]. Clearly, $W \notin \Phi$. This shows that $\Phi$ does not have a minimal element. By Zorn’s lemma, therefore, $\Phi$ is empty.

Consider the topology on the category of schemes generated by families of étale maps that are jointly surjective on $k$-points for every field $k$. The previous observation shows that this topology is generated by open covers and étale maps admitting finitely presented splitting sequences. In other words, the topology originally defined by Nisnevich in [Nis89] coincides with the Nisnevich topology defined in [Hoy14, Appendix C], despite an unsubstantiated claim to the contrary in loc. cit. By [AHW15, Proposition 2.3.2], this topology is even generated by open covers and maps of the form $\text{Spec}(B \times A_f) \to \text{Spec}(A)$, where $A \to B$ is étale and induces an isomorphism $A/f \simeq B/f$.

References


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