ON QUILLEN'S PLUS CONSTRUCTION

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Abstract. A discussion of Quillen’s plus construction from an $\infty$-categorical perspective.

Let $\mathcal{X}$ be an $\infty$-topos. An object $X \in \mathcal{X}$ is called acyclic if the map $X \to *$ is an epimorphism in the categorical sense, i.e., if the square

\[
\begin{array}{ccc}
X & \to & * \\
\downarrow & & \downarrow \\
* & \to & *
\end{array}
\]

is cocartesian. Note that when $\mathcal{X} = \mathcal{S}$, this is equivalent to $\check{H}_* (X, \mathbb{Z}) = 0$. We shall say that a map $X \to Y$ in $\mathcal{X}$ is acyclic if it is acyclic as an object of $\mathcal{X} / Y$. The class of acyclic maps is closed under composition, base change, cofiber change, colimits, and finite products. Moreover, if $g \circ f$ and $f$ are acyclic, then $g$ is acyclic.

Lemma 1. Acyclic maps form the left class of a modality on $\mathcal{X}$.

Proof. It suffices to show that the class of acyclic maps is of small generation as a saturated class. The full subcategory of $\mathcal{X}$ spanned by the acyclic objects is accessible, being the fiber of the suspension functor. It is thus generated under filtered colimits by a small subcategory. Let $\mathcal{C} \subset \text{Fun}(\Delta^1, \mathcal{X})$ be the union of these small subcategories of $\mathcal{X} / \mathcal{X}$ as $X$ ranges over a small set of generators of $\mathcal{X}$. Using that acyclic maps are stable under base change, we immediately deduce that $\mathcal{C}$ generates the class of acyclic maps under colimits.

In particular, every morphism $f$ in $\mathcal{X}$ factors uniquely as $f = h \circ g$ where $g$ is acyclic and $h$ is right orthogonal to acyclic maps. The plus construction $X \mapsto X^+$ is the localization functor associated with this factorization system, i.e., $X \to X^+$ is the acyclic map such that $X^+$ is local with respect to acyclic maps.

For $X \in \mathcal{X}$, recall that $\pi_n (X)$ is a discrete object in $\mathcal{X} / \mathcal{X}$, which is a group if $n \geq 1$ (abelian if $n \geq 2$).

Lemma 2 (Hurewicz theorem). Let $X \in \mathcal{X}$ be an $n$-connective object for some $n \geq 1$. Then the Hurewicz map $\pi_n (X) \to H_n (X, \mathbb{Z}) \times X$ in $\mathcal{X} / \mathcal{X}$ exhibits $H_n (X, \mathbb{Z}) \times X$ as the abelianization of $\pi_n (X)$.

Proof. If $\mathcal{X}$ is a presheaf $\infty$-topos, this follows from the classical Hurewicz theorem. If $f_* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism and the result holds for some $n$-connective object $Y \in \mathcal{Y}$, then the result holds for $f^* (Y)$. It remains to observe that $X$ is the preimage by a geometric morphism of an $n$-connective object in a presheaf $\infty$-topos $\mathcal{Y}$. Indeed, if $g_* : \mathcal{X} \to \mathcal{P} (\mathcal{C})$ is a fully faithful geometric morphism, one can take $\mathcal{Y} = \mathcal{P} (\mathcal{C}) / \tau_{\leq n-1} (\mathcal{C}) \simeq \mathcal{P} (\mathcal{C} / \tau_{\leq n-1} (\mathcal{C}))$.

Recall that a discrete group is perfect if its abelianization is trivial, and hypoabelian if it has no nontrivial perfect subgroups.

Lemma 3. Let $X \in \mathcal{X}$ be acyclic. Then $X$ is 1-connective and $\pi_1 (X)$ is perfect. If $\pi_1 (X)$ is trivial, then $X$ is $\infty$-connective.

Proof. If $\mathcal{C}$ is stable and $F : \mathcal{X} \to \mathcal{C}$ preserves pushouts, then clearly $F (X) \simeq F (\ast)$. In particular, $\check{H}_0 (X, \mathbb{Z})$ and $H_1 (X, \mathbb{Z})$ are trivial. The former implies that $X$ is 1-connective. By the latter and the Hurewicz theorem, $\pi_1 (X)$ is perfect. The final statement follows immediately from the Blakers–Massey theorem.

Remark 4. We do not know an example of an $\infty$-connective acyclic object that is not contractible.

Corollary 5. Let $f : X \to Y$ be an acyclic morphism in $\mathcal{X}$. If $\pi_1 (X)$ is hypoabelian, then $f$ is $\infty$-connective. In particular, if $\mathcal{X}$ is moreover hypercomplete, then $X \simeq X^+$.

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Proof. It follows from the exact sequence

\[ f^* \pi_2(Y) \to \pi_1(f) \to \pi_1(X) \]

that \( \pi_1(f) \) is hypoabelian. By Lemma 3, we conclude that \( \pi_1(f) \) is trivial, hence that \( f \) is \( \infty \)-connective. \( \square \)

**Lemma 6** (van Kampen theorem). Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g'} \\
Z & \xrightarrow{f'} & W
\end{array}
\]

be a pushout square in \( \mathcal{X} \) where \( f \) and \( g \) induce isomorphisms on \( \tau_{\leq 0} \) and let \( h = g' \circ f \). Then

\[
\begin{array}{ccc}
\pi_1(X) & \xrightarrow{f^*} & \pi_1(Y) \\
\downarrow & & \downarrow \\
g^* \pi_1(Z) & \xrightarrow{h^*} & \pi_1(W)
\end{array}
\]

is a pushout square of groups.

Proof. Replacing \( X \) by \( X_{1/\tau_{\leq 0}} \), we may assume that \( X, Y, Z, \) and hence \( W \) are 1-connective. As \( X \) is in particular 0-connective, we may assume that it has a global section \( s : * \to X \). Then \( \tau_{\leq 1} X \simeq B s^* \pi_1(X) \), and similarly for \( Y, Z, \) and \( W \). Since \( B \) induces an equivalence of categories between discrete groups and pointed 1-connective 1-truncated objects, we deduce that \( s^* \) of the given square of groups is a pushout square. But \( s \) is 0-connective, so this suffices. \( \square \)

**Lemma 7.** Let \( \mathcal{X} \) be a hypercomplete \( \infty \)-topos, let \( f : X \to Y \) be an acyclic morphism in \( \mathcal{X} \), and let \( P \) be the kernel of \( \pi_1(X) \to f^* \pi_1(Y) \). Then \( f \) is the initial morphism that kills \( P \).

Proof. Let \( g : X \to X' \) be a morphism that kills \( P \), and consider the pushout square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g'} \\
X' & \xrightarrow{f'} & Y'.
\end{array}
\]

Then \( f' \) is acyclic, and we must show that it is an equivalence. Write \( g = g'' \circ g' \) where \( g' \) is 1-connective and \( g'' \) is 0-truncated. Then \( g' \) still kills \( P \), so we can assume that \( g \) is 1-connective. Since \( f \) is acyclic, it is 1-connective by Lemma 3. By the van Kampen theorem, the associated square of groups

\[
\begin{array}{ccc}
\pi_1(X) & \xrightarrow{f^*} & \pi_1(Y) \\
\downarrow & & \downarrow \\
g^* \pi_1(X') & \xrightarrow{g'^*} & \pi_1(Y')
\end{array}
\]

is a pushout square. It follows that the lower horizontal map is an isomorphism, so that \( g^* \pi_1(f') \) is abelian. It is also perfect by Lemma 3, hence trivial. Since \( g \) is 0-connective, \( \pi_1(f') \) is trivial. By Lemma 3, we deduce that \( f' \) is \( \infty \)-connective, whence an equivalence. \( \square \)

Let \( \mathcal{X}^{\otimes} \) be the \( \infty \)-category of pairs \((X, P)\) where \( X \in \mathcal{X} \) and \( P \) is a perfect subgroup of \( \pi_1(X) \); a morphism \((X, P) \to (Y, Q)\) is a morphism \( f : X \to Y \) sending \( P \) to \( Q \).

We say that \( \pi_1 \) preserves products if, for every family of objects \((X_\alpha)_\alpha\) with product \( X \), the canonical map \( \pi_1(X) \to \prod_\alpha p^*_\alpha \pi_1(X_\alpha) \) is an equivalence, where \( p_\alpha : X \to X_\alpha \) is the projection. This condition holds in any presheaf \( \infty \)-topos, and is inherited by essential subtopoi.

**Theorem 8.** Let \( \mathcal{X} \) be an \( \infty \)-topos where \( \pi_1 \) preserves products. Then the fully faithful functor \( \mathcal{X} \hookrightarrow \mathcal{X}^{\otimes}, \ X \mapsto (X, 1) \), admits a left adjoint \((X, P) \mapsto X/P\). Moreover, the unit map \( \eta : X \to X/P \) is acyclic.
Proof. Fix \((X, P) \in \mathcal{X}^\circ\), and let \(\mathcal{X}_{(X, P)}\) be the full subcategory of \(\mathcal{X}_X\) consisting of the morphisms \(f: X \to Y\) that kill \(P\). Clearly, \(\mathcal{X}_{(X, P)}\) is accessible, and in particular it has a small coinitial subcategory. It therefore suffices to show that \(\mathcal{X}_{(X, P)}\) is closed under nonempty limits in \(\mathcal{X}_X\). The assumption that \(\pi_1\) preserves products implies that \(\mathcal{X}_{(X, P)}\) is closed under products. It remains to show that \(\mathcal{X}_{(X, P)}\) is closed under pullbacks. Given a cartesian square

\[
\begin{array}{ccc}
Y & \longrightarrow & Y_0 \\
\downarrow & & \downarrow \\
Y_1 & \longrightarrow & Y_{01}
\end{array}
\]

under \(X\) such that \(P\) is killed in \(\pi_1(Y_0)\) and \(\pi_1(Y_1)\), we must show that \(P\) is killed in \(\pi_1(Y)\). The exact sequence

\[
\pi_2(Y_0) \to \pi_1(Y) \to \pi_1(Y_0) \times \pi_1(Y_1)
\]

in \(\mathcal{X}_Y\) shows that the image of \(P\) in \(\pi_1(Y)\) is abelian, hence trivial since \(P\) is perfect. The fact that \(\eta\) is acyclic follows immediately from its universal property. \(\square\)

Remark 9. In the context of Theorem 8, there is an epimorphism \(\pi_1(X) \to \eta^* \pi_1(X/P)\) whose kernel is perfect and contains \(P\). We do not know if its kernel equals \(P\) in general. This is equivalent to the existence of a morphism \(f: X \to Y\) that kills exactly \(P\). If \(\mathcal{X} = \mathcal{S}\), it is trivial to construct such a morphism where \(Y\) is a groupoid. In that case, \(X \to X/P\) is an acyclic map that kills exactly \(P\), which is one of the standard characterizations of Quillen’s plus construction. It follows that \(X \to X/P\) kills exactly \(P\) whenever the 2-topos \(\mathcal{X}_{\leq 1}\) has enough points.

We will say that \(X \in \mathcal{X}\) is hypoabelian if the group \(\pi_1(X)\) is hypoabelian. As always, a morphism \(f: X \to Y\) in \(\mathcal{X}\) is hypoabelian if it is so as an object of \(\mathcal{X}_Y\). Since \(\pi_1(f)\) is an extension of the kernel of \(\pi_1(X) \to f^* \pi_1(Y)\) by a quotient of \(f^* \pi_2(Y)\), \(f\) is hypoabelian if and only if that kernel is hypoabelian.

Corollary 10. Let \(\mathcal{X}\) be a hypercomplete \(\infty\)-topos where \(\pi_1\) preserves products. Then a morphism in \(\mathcal{X}\) is right orthogonal to acyclic morphisms if and only if it is hypoabelian. Hence, for any \(X \in \mathcal{X}\), we have \(X^+ \simeq X/P\) where \(P \subset \pi_1(X)\) is the maximal perfect subgroup.

Proof. One implication was already proved in Corollary 5. Suppose \(f: X \to Y\) is right orthogonal to acyclic maps. Let \(K\) be the kernel of \(\pi_1(X) \to f^* \pi_1(Y)\) and let \(P \subset K\) be a perfect subgroup. Then \(f\) factors uniquely as \(X \xrightarrow{\eta} X/P \xrightarrow{\delta} Y\). Since \(\eta\) is acyclic and \(f\) is right orthogonal to acyclic maps, \(\eta\) admits a retraction, hence \(\pi_1(X)\) is a retract of \(\eta^* \pi_1(X/P)\), hence \(P = 1\). \(\square\)

For an arbitrary \(\infty\)-topos \(\mathcal{X}\), the proof of Theorem 8 shows that \(\mathcal{X}_{(X, P)}\) is closed under nonempty finite limits in \(\mathcal{X}_X\), so the inclusion \(\mathcal{X} \hookrightarrow \mathcal{X}^\circ\) admits a left adjoint \(X^\circ \to \text{Pro}(\mathcal{X})\), \((X, P) \mapsto X/P\). Moreover, if \(f^*_\ast: Y \to X\) is a geometric morphism of \(\infty\)-topoi and \((X, P) \in \mathcal{X}^\circ\), we have \(f^*(X/P) \simeq f^*(X)/f^*(P)\) by comparison of universal properties. For example, suppose \(\mathcal{X}\) is a subtopos of a presheaf \(\infty\)-topos \(\mathcal{P}(\mathcal{C})\), and let \(a: \mathcal{P}(\mathcal{C}) \to \mathcal{X}\) be the left adjoint to the inclusion. If \(X \in \mathcal{X}\) and \(P\) is a perfect subgroup of the fundamental group of \(X\) in \(\mathcal{P}(\mathcal{C})\), then \(X/a(P)\) exists in \(\mathcal{X}\), since \(X/a(P) \simeq a(X/P)\).

Example 11. Let \(\mathcal{C}\) be some category of qcqs schemes equipped with the Zariski topology. Let \(\text{Vect}(X)\) be the groupoid of finite locally free sheaves on \(X\), and let \(\text{sVect}(X)\) be the colimit of the sequence

\[
\text{Vect}(X) \leftarrow \text{Vect}(X) \leftarrow \cdots
\]

Then \(\text{sVect} \in \text{Shv}(\mathcal{C})\) and \(\pi_1(\text{sVect})(X)\) is the group \(\text{GL}(\xi)\) of automorphisms of \(\xi \in \text{sVect}(X)\). Let \(\text{SL} \subset \text{GL}\) be the subgroup of automorphisms of determinant 1, which is the subsheaf of \(\text{GL}\) generated by elementary matrices. Then \(\text{SL}\) is a perfect subgroup of \(\pi_1(\text{sVect})\) and

\[\text{sVect}/\text{SL} \simeq K\]

in \(\text{Shv}(\mathcal{C})\), where \(K\) is Thomason–Trobaugh K-theory. In other words, if \(F: \mathcal{C}^\text{op} \to \mathcal{S}\) is a Zariski sheaf and \(f: \text{sVect} \to F\) is a map that kills \(\text{SL}\) (e.g., \(\pi_1(F)\) is abelian or \(F\) is \(K\)-homotopy invariant), then \(f\) factors uniquely through \(K\).

We conclude with a version of the McDuff–Segal group completion theorem.
Theorem 12. Let $\mathcal{X}$ be an $\infty$-topos, $M = \coprod_{n \geq 0} M_n$ an $\mathbb{N}$-graded homotopy-commutative monoid in $\mathcal{X}$ with $\tau_{\leq 0}(M) \simeq \mathbb{N}$, and $x : * \to M_1$ a global section. Let $M^{gp}$ be the group completion of $M$, let

$$M_\infty = \text{colim}(M_0 \xrightarrow{x} M_1 \xrightarrow{x} M_2 \to \cdots),$$

and let $P \subset \pi_1(M_\infty)$ be the commutator subgroup. Then $P$ is perfect and the canonical map $\mathbb{Z} \times M_\infty \to M^{gp}$ induces an equivalence $\mathbb{Z} \times M_\infty/P \simeq M^{gp}$. In particular, $\mathbb{Z} \times M_\infty \to M^{gp}$ is acyclic.

Proof. When $\mathcal{X} = S$, the classical group completion theorem states that $\mathbb{Z} \times M_\infty \to M^{gp}$ is acyclic. Since $\pi_1(M^{gp})$ is abelian, this implies that $P$ is perfect and that $\mathbb{Z} \times M_\infty/P \simeq M^{gp}$ (by Lemma 7). This immediately generalizes to the case of a presheaf $\infty$-topos. In general, suppose that $\mathcal{X}$ is a subtopos of $\mathcal{P}(\mathcal{C})$, and let $a : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ be the left adjoint to the inclusion. The graded pieces $M_n$ assemble into an $\mathbb{N}$-graded monoid $M'$ in $\mathcal{P}(\mathcal{C})$ such that $a(M') \simeq M$. The section $x$ defines a morphism of $\mathbb{N}$-graded monoids $\mathbb{N} \to M'$, and we let $M'' \subset M'$ be its image. Then $a(M'') \simeq M$ and $M''$ satisfies the assumptions of the theorem in $\mathcal{P}(\mathcal{C})$. The result for $M$ in $\mathcal{X}$ then follows from the result for $M''$ in $\mathcal{P}(\mathcal{C})$.  \qed