

Towards higher algebraic K-theory

Idea: replace groupoids by ∞ -groupoids.

What are ∞ -groupoids?

A 0-groupoid is a set

A 1-groupoid is a groupoid.

A 2-groupoid has objects, $\underset{\text{iso}}{\text{morphisms}}$ between objects, $\text{isomorphisms between isos.}$ (2-isomorphisms).

e.g.: groupoids form a 2-groupoid

A 3-groupoid has 3-isos between 2-isos., etc...

\vdots

∞ -groupoid

Naive definition: \bullet a strict n -category is a $\overset{\text{strict}}{\text{category}}$ enriched in $(n-1)$ -categories
 \bullet a strict n -groupoid is a strict n -category where all i -morphisms are isomorphisms ($1 \leq i \leq n$)

Problem: this breaks the principle of equivalence.

in a strict 2-category: $(f \circ g) \circ h \cong f \circ (g \circ h) \rightsquigarrow$ need pentagon arrow...

equivalent to strict n -cat $\left\{ \begin{array}{l} 1\text{-category: can define in a few lines (Eilenberg-Mac Lane).} \\ 2\text{-category: Bezkotevich} \sim 1-2 \text{ pages} \end{array} \right.$

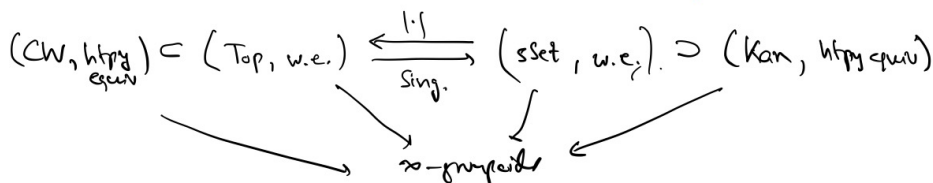
not equiv to strict n -cat. $\left\{ \begin{array}{l} 3\text{-categories: Gordon-Power-Street} \quad 6 \text{ pages.} \\ 4\text{-category: Trimble} \quad 51 \text{ pages.} \end{array} \right.$

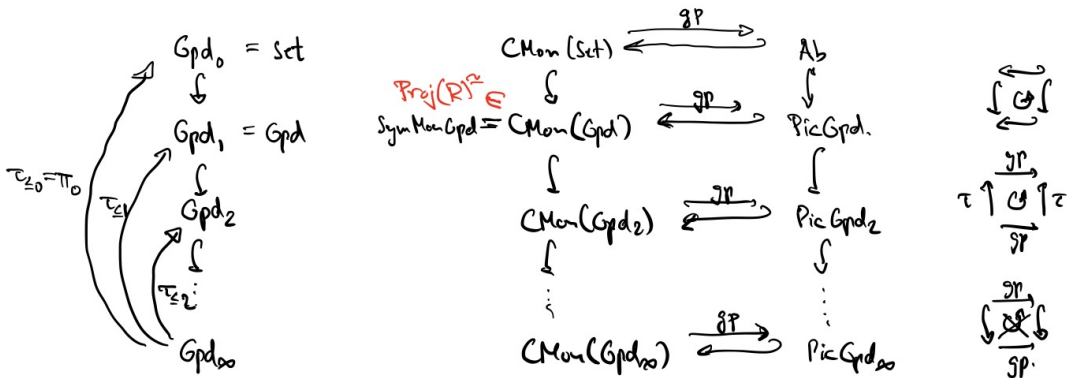
Homotopy hypothesis (Grothendieck)

$$\{\infty\text{-groupoids}\} \cong \text{Top} [\text{weak equivalences}^{-1}]$$

$$\{n\text{-groupoids}\} \cong \text{Top}_{\leq n} [\text{w.e.}^{-1}]$$

$\uparrow \pi_i = 0 \text{ for } i > n$





Simplicial sets

Def. The simplex category Δ is the category of nonempty finite ordered sets (morphisms are monotone maps).

Write $[n] = \{0 < 1 < \dots < n\}$

So Δ has objects $[0], [1], [2], \dots$

A simplicial object in a category \mathcal{C} is a functor $\Delta^{op} \rightarrow \mathcal{C}$.

Notation: $s\mathcal{C} = \text{Fun}(\Delta^{op}, \mathcal{C})$.

Examples: • Δ^n is the simplicial set $\Delta^n([m]) = \text{Hom}_{\Delta}([m], [n])$

In other words $[n] \mapsto \Delta^n$ is the Yoneda embedding $\Delta \hookrightarrow s\text{Set}$

• $\partial\Delta^n \subset \Delta^n$ $\partial\Delta^n([m]) \subset \text{Hom}_{\Delta}([m], [n])$ non-surjective maps.

• $0 \leq k \leq n$, $\Lambda_k^n \subset \partial\Delta^n$ is defined by:

$$\Lambda_k^n([m]) \subset \text{Hom}_{\Delta}([m], [n])$$

$$f \in \Lambda_k^n([m]) \Leftrightarrow [n] - \{k\} \neq \text{image of } f.$$

Presentation of Δ : $\delta_i: [n] \hookrightarrow [n+1]$ $0 \leq i \leq n$, skips i

$\sigma_i: [n] \rightarrow [n-1]$ $0 \leq i \leq n-1$, hits i twice

$$X: \Delta^{op} \rightarrow \mathcal{C}, \quad X_n = X([n]) \quad d_i = \delta_i^*: X_{n+1} \rightarrow X_n$$

$$s_i = \sigma_i^*: X_{n-1} \rightarrow X_n$$

$$\text{Fun}(\Delta^{op}, \mathcal{C}) \cong \left\{ \begin{array}{l} X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xrightarrow{d_1} \end{array} X_1 \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xrightarrow{s_1} \\ \xrightarrow{d_2} \end{array} X_2 \begin{array}{c} \xrightarrow{\dots} \\ \xrightarrow{\dots} \\ \xrightarrow{\dots} \end{array} X_3 \dots \end{array} \right. \quad \text{s.t.} \quad \left. \begin{array}{l} d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases} \\ d_i \circ d_j = d_{j+1} \circ d_i \quad \text{if } i < j \\ s_i \circ s_j = s_j \circ s_{i-1} \quad \text{if } i > j \end{array} \right\}$$

Def. The maps $d_i: X_{n-1} \rightarrow X_n$ are called face maps
 $s_i: X_{n-1} \rightarrow X_n$ are called degeneracy maps.
The elements of X_n are called n-simplices
 $n=0$: vertices
 $n=1$: edges

An n -simplex is degenerate if it is in the image of a degeneracy map.
A simplicial set is finite if it has finitely many non-degenerate simplices.