Towards higher algebraic K-theory

Idea: replace groupoids by ∞-groupoids.

What are ∞-groupoids?

A 0-groupoid is a set
A 1-groupoid is a groupoid.
A 2-groupoid has objects, iso-morphisms between objects, isomorphisms between isos.
(2-isomorphisms).

e.g.: groupoids form a 2-groupoid

A 3-groupoid has 3-isos between 2-isos., etc...
$\vdots$
∞-groupoid

Naive definition:
- a strict n-category is a category enriched in strict $(n-1)$-categories
- a strict n-groupoid is a strict n-category where all i-morphisms are isomorphisms $(1 \leq i \leq n)$

Problem: this breaks the principle of equivalence.

in a strict 2-category: $(f \circ g) \circ h \cong f \circ (g \circ h)$ $\rightsquigarrow$ need pentagon axiom...

equivalent to strict n-cat $\left\{\begin{array}{lll} \text{1-category:} & \text{can define in a few lines} & \text{(Eilenberg–MacLane).} \\ \text{2-category:} & \text{Bénabou} & \sim 1\text{-}2 \text{ pages} \end{array}\right.$

not equiv to strict n-cat. $\left\{\begin{array}{lll} \text{3-categories:} & \text{Gordon–Power–Street} & \text{6 pages.} \\ \text{4-category:} & \text{Trimble} & \text{51 pages.} \end{array}\right.$

Homotopy hypothesis (Grothendieck)

$$\{\infty\text{-groupoids}\} \simeq \mathrm{Top}[\underbrace{\text{weak equivalences}}_{\text{iso on } \pi_*}{}^{-1}]$$

$$\{n\text{-groupoids}\} \simeq \underbrace{\mathrm{Top}_{\leq n}}_{\pi_i = 0 \text{ for } i > n}[\text{w.e.}^{-1}]$$

$$\begin{tikzcd}
(\mathrm{CW}, \text{htpy equiv}) \arrow[r, phantom, "\subset"] \arrow[drr] & (\mathrm{Top}, \text{w.e.}) \arrow[rr, shift right, "\mathrm{Sing.}"'] \arrow[dr] & & (\mathrm{sSet}, \text{w.e.}) \arrow[ll, shift right, "|\cdot|"'] \arrow[dl] \arrow[r, phantom, "\supset"] & (\mathrm{Kan}, \text{htpy equiv}) \arrow[dll] \\
& & \infty\text{-groupoids} & &
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Gpd}_0 = \mathrm{Set} \arrow[d, hook] & & \mathrm{CMon}(\mathrm{Set}) \arrow[r, shift left, "gp"] \arrow[d, hook] & \mathrm{Ab} \arrow[l, shift left, hook] \arrow[d, hook] \\
\mathrm{Gpd}_1 = \mathrm{Gpd} \arrow[d, hook] & \color{red}{\mathrm{Proj}(R)^{\simeq} \in}\ \mathrm{SymMonGpd} = & \mathrm{CMon}(\mathrm{Gpd}) \arrow[r, shift left, "gp"] \arrow[d, hook] & \mathrm{PicGpd} \arrow[l, shift left, hook] \arrow[d, hook] \\
\mathrm{Gpd}_2 \arrow[d, hook] & & \mathrm{CMon}(\mathrm{Gpd}_2) \arrow[r, shift left, "gp"] \arrow[d, hook] & \mathrm{PicGpd}_2 \arrow[l, shift left, hook] \arrow[d, hook] \\
\vdots \arrow[d, hook] & & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] \\
\mathrm{Gpd}_\infty \arrow[uuuu, bend left, "\tau_{\leq 0} = \pi_0"] \arrow[uuu, bend left, "\tau_{\leq 1}"] \arrow[uu, bend left, "\tau_{\leq 2}"'] & & \mathrm{CMon}(\mathrm{Gpd}_\infty) \arrow[r, shift left, "gp"] & \mathrm{PicGpd}_\infty \arrow[l, shift left, hook]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, hook] \\
\cdot & \cdot \arrow[l, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "gp"] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \\
\cdot \arrow[u, "\tau"] \arrow[r, "gp"] & \cdot \arrow[u, "\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "gp"] \arrow[d, hook] \arrow[dr, phantom, "\cancel{\circlearrowleft}"] & \cdot \arrow[d, hook] \\
\cdot \arrow[r, "gp."] & \cdot
\end{tikzcd}
$$

Simplicial sets

Def. The simplex category Δ is the category of nonempty finite ordered sets (morphisms are monotone maps).

Write $[n] = \{0 < 1 < \cdots < n\}$

So Δ has objects $[0], [1], [2], \ldots$

A simplicial object in a category C is a functor $\Delta^{op} \longrightarrow C$.

Notation: $sC = \mathrm{Fun}(\Delta^{op}, C)$.

Examples:
- Δ^n is the simplicial set $\Delta^n([m]) = \mathrm{Hom}_\Delta([m],[n])$

  In other words $[n] \mapsto \Delta^n$ is the Yoneda embedding $\Delta \hookrightarrow s\mathrm{Set}$
- $\partial\Delta^n \subset \Delta^n$ $\quad \partial\Delta^n([m]) \subset \mathrm{Hom}_\Delta([m],[n])$ non-surjective maps.
- $0 \leq k \leq n$, $\Lambda^n_k \subset \partial\Delta^n$ is defined by:

$$\Lambda^n_k([m]) \subset \mathrm{Hom}_\Delta([m],[n])$$

$$f \in \Lambda^n_k([m]) \iff [n] \setminus \{k\} \not\subset \text{image of } f.$$

Presentation of Δ:
$\delta_i : [n] \hookrightarrow [n+1]$ $\quad 0 \leq i \leq n+1$, skips i

$\sigma_i : [n] \twoheadrightarrow [n-1]$ $\quad 0 \leq i \leq n-1$, hits i twice

$X : \Delta^{op} \to C$, $\quad X_n = X([n])$ $\qquad d_i = \delta_i^* : X_{n+1} \longrightarrow X_n$

$s_i = \sigma_i^* : X_{n-1} \longrightarrow X_n$

$$
\mathrm{Fun}(\Delta^{op}, C) \simeq \left\{ X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} \overset{s_0}{\rightarrow} X_1 \underset{s_1}{\overset{s_0}{\rightrightarrows}} \underset{d_2}{\overset{d_0}{\leftleftarrows}} X_2 \equiv X_3 \cdots \quad \text{s.t.} \quad
\begin{array}{l}
d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i<j \\ \mathrm{id} & \text{if } i=j \text{ or } j+1 \\ s_j \circ d_{i-1} & \text{if } i>j+1 \end{cases} \\
d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if } i<j \\
s_i \circ s_j = s_j \circ s_{i-1} \quad \text{if } i>j
\end{array}
\right\}
$$

<u>Def.</u> The maps $d_i : X_{n+1} \to X_n$ are called <u>face maps</u>

$s_i : X_{n-1} \to X_n$ are called <u>degeneracy maps</u>.

The elements of X_n are called <u>n-simplices</u>

$n=0$: <u>vertices</u>

$n=1$: <u>edges</u>

An n-simplex is <u>degenerate</u> if it is in the image of a degeneracy map.

A simplicial set is <u>finite</u> if it has finitely many non-degenerate simplices.