Remark More generally, for any normal perfect subgroup $P \subset \pi_1(X,x)$, there is a universal map $X \longrightarrow X_P^+$ that kills P, and it is acyclic.
Consequently, if X is connected, there is an equivalence of categories:

$$\mathrm{Acyclic}_{X/} \subset \mathrm{Top}[w.e.^{-1}]_{X/}$$

$$X \to X_P^+ \;\overset{\simeq}{\longleftrightarrow}\; \ker(\pi_1)$$

$$P \in \left\{ \begin{array}{l} \text{normal perfect} \\ \text{subgroups of } \pi_1(X,x) \end{array} \right\}.$$

$$\left(\begin{tikzcd} X \arrow[r, "\text{acyc}", "f"'] \arrow[dr] & Y \\ & X^+_{\ker(\pi_1(f))} \arrow[u, dashed] \end{tikzcd} \quad \exists! \text{ acyclic and iso on } \pi_1 \overset{\text{Whitehead}}{\Longrightarrow} \text{w.e.} \right)$$

$$\begin{tikzcd} X \times X \arrow[rr, "\text{swap}"] \arrow[dr, "\mu"'] & \circlearrowleft & X \times X \arrow[dl, "\mu"] \\ & X & \end{tikzcd} \quad \text{in } \mathrm{Top}[w.e.^{-1}]$$

Group completion theorem (McDuff–Segal, Randal-Williams)

Let X be an E_∞-space (more generally, a homotopy commutative E_1-space).
Suppose there is a cofinal embedding $\mathbb{N} \subset \pi_0(X)$. Let

$$X_\infty = \operatorname{colim}\left(X \xrightarrow{+1} X \xrightarrow{+1} X \xrightarrow{+1} \cdots \right)$$

$\mu(1,-) : X \to X$.

Then $X^{gp} \simeq X_\infty^+$.
Moreover, $\pi_1(X_\infty^+) = \pi_1(X_\infty)^{ab}$.

Idea of proof From the universal property of X^{gp}, one can show that

$$H_*(X^{gp}) \simeq H_*(X)[s^{-1}] \quad \text{where} \quad \begin{array}{ccc} \mathbb{N} \subset \pi_0(X) & \longrightarrow & H_0(X) \\ 1 & \longmapsto & s \end{array}$$

$$= \operatorname{colim}\left(H_*(X) \xrightarrow{\cdot s} H_*(X) \xrightarrow{\cdot s} \cdots \right) = H_*(X_\infty)$$

$\Longrightarrow$ the canonical map $X_\infty \longrightarrow X^{gp}$ is an isomorphism on $H_*(-)$.

"With more work", one can show $X_\infty \longrightarrow X^{gp}$ is acyclic.

By Eckmann–Hilton $\pi_1(X^{gp})$ are abelian $\Longrightarrow$ $\exists$ $X_\infty^+ \longrightarrow X^{gp}$

This map is acyclic, and iso on π_1 because:

$$\begin{tikzcd} \pi_1(X_\infty) \arrow[r, two heads] \arrow[d, two heads] & \pi_1(X^{gp}) \\ \pi_1(X_\infty^+) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

$\blacksquare$

Corollary Let R be a ring. Then $K(R) \simeq K_0(R) \times BGL(R)^+$.

Pf. Apply the group completion theorem to $X = |N(\mathrm{Proj}(R)^{\simeq})|$

$$X_\infty = \mathrm{colim}\left(X \xrightarrow{\oplus R} X \xrightarrow{\oplus R} X \to \cdots\right)$$

$$\pi_0(X_\infty) = \mathrm{colim}\left(\pi_0 X \xrightarrow{\oplus R} \pi_0 X \to \cdots\right) = \pi_0(X)^{gp} = K_0(R)$$

$P \in \mathrm{Proj}(R)$, $\pi_1(X_\infty, P) = \mathrm{colim}\left(\mathrm{Aut}_R(P) \xrightarrow{\oplus R^m} \mathrm{Aut}_R(P \oplus R^m) \xrightarrow{\oplus R^m} \mathrm{Aut}_R(P \oplus R^{2m}) \to \cdots\right)$

Choose Q s.t. $P \oplus Q \simeq R^m$

$$\begin{tikzcd}
\mathrm{Aut}_R(P) \arrow[r, "\oplus R^m"] \arrow[d, "\oplus Q"] & \mathrm{Aut}_R(P \oplus R^m) \arrow[r, "\oplus R^m"] \arrow[d, "\oplus Q"] & \mathrm{Aut}_R(P \oplus R^{2m}) \arrow[r] \arrow[d, "\oplus Q"] & \cdots \\
\mathrm{Aut}_R(R^m) \arrow[r, "\oplus R^m"] \arrow[ur, "\oplus P"] & \mathrm{Aut}_R(R^{2m}) \arrow[r, "\oplus R^m"] \arrow[ur, "\oplus P"] & \mathrm{Aut}_R(R^{3m}) \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

$$\Rightarrow \pi_1(X_\infty, P) \simeq GL(R)$$

$$\Rightarrow X_\infty \simeq \coprod_{K_0(R)} BGL(R) = K_0(R) \times BGL(R)$$

$$\Rightarrow K(R) \simeq X_\infty^+ = K_0(R) \times BGL(R)^+ . \qquad \blacksquare$$

Remark $\pi_1(BGL(R)^+) = \pi_1(BGL(R)) / \text{max perfect subgroup}$

The max perfect subgroup of $GL(R)$ is $E(R)$ (because $E(R) = [GL(R), GL(R)]$ and is perfect)

So this recovers $K_1(R) = GL(R)/E(R)$.

Remark (Connection w/ classical K_2).

If G is a group and $P \subset G$ maximal perfect subgroup, then $\pi_2(BG^+)$ is the center of the universal central extension of P.

Let F be the homotopy fiber of $BG \to BG^+$.

$$\begin{tikzcd}
\overset{0}{\underset{\shortparallel}{}}\pi_2(BG) \arrow[r] & \underbrace{\pi_2(BG^+) \arrow[r] & \pi_1(F) \arrow[r] \arrow[dr]}_{\substack{\text{universal} \\ \text{central extension} \\ \text{of } P.}} & \overset{G}{\underset{\shortparallel}{}}\pi_1(BG) \arrow[r, two heads] & \overset{G/P}{\underset{\shortparallel}{}}\pi_1(BG^+) \arrow[r] & \overset{*}{\underset{\shortparallel}{}}\pi_0(F) \\
 & & & P. \arrow[u] & &
\end{tikzcd}$$

In particular, $\pi_2(BGL(R)^+) = \ker(St(R) \twoheadrightarrow E(R))$

(Milnor's definition of $K_2(R)$).

(Weibel, Ch. IV, Prop 1.7)

Corollary $|N(\mathrm{Fin}^{\simeq})|^{gp} \simeq \mathbb{Z} \times B\Sigma_\infty^+$.