Monoids revisited

Comm. monoids in a cat. C : M with $M \times M \xrightarrow{\mu} M$ binary operation

$* \xrightarrow{e} M$ nullary operation

+ axioms

The axioms ensure that for any finite set I, there is a well-defined multiplication map $\mu_I : M^I \longrightarrow M$.

$$J = \coprod_{i \in I} J_i \rightsquigarrow \begin{tikzcd} M^J \simeq \prod_{i\in I} M^{J_i} \arrow[r, "\prod \mu_{J_i}"] \arrow[d, "\mu_J"'] \arrow[dr, phantom, "\circlearrowleft"] & \prod_{i\in I} M \arrow[d, equal] \\ M & M^I \arrow[l, "\mu_I"'] \end{tikzcd}$$

$\Updownarrow$

$J \xrightarrow{f} I \quad J_i = f^{-1}(i)$

for any $f: J \to I$, we have $\mu_f : M^J \longrightarrow M^I$

$$K \xrightarrow{g} J \xrightarrow{f} I \rightsquigarrow \mu_f \circ \mu_g = \mu_{f \circ g}$$

The comm. monoid structure on M defines a functor
$$\begin{aligned} \mu : \mathrm{Fin} &\longrightarrow C \\ I &\longmapsto M^I \\ (f: J \to I) &\longmapsto \mu_f . \end{aligned}$$

<u>Q:</u> Can we recover M from the functor μ?

Not quite...

We also need isomorphisms $\mu(I) \simeq \mu(*)^I \xrightarrow{\pi_i} \mu(*)$ $\quad i \in I$

We need to add morphisms to Fin :

$\mathrm{Fin} \subset \mathrm{Fin}' \cong \Gamma^{op}$ (Segal)

Fin' : objects are finite sets, morphisms are partially defined maps

- A part. defined map from X to Y is a pair (D, f) where $D \subset X$ and $f: D \longrightarrow Y$

Composition:

$$\begin{tikzcd} X & Y & Z \\ D \arrow[u, phantom, "\cup" description] \arrow[ur, "f"] & E \arrow[u, phantom, "\cup" description] \arrow[ur, "g"] & \\ f^{-1}(E) \arrow[u, phantom, "\cup" description] \arrow[ur] & & \end{tikzcd}$$

Remark: $\mathrm{Fin}' \simeq \mathrm{Fin}_*$

$$\begin{tikzcd} A \arrow[r, phantom, "\supset"] & f^{-1}(B) \arrow[dl, "f"] & A_+ = A \cup \{*\} \arrow[l, mapsto] \arrow[d, "f"] \\ B & & B_+ \end{tikzcd}$$

Given $J \supset D \xrightarrow[f]{} I$, we get $M^J \xrightarrow{\mathrm{res}} M^D \xrightarrow{\mu_f} M^I$

So a commutative monoid M in C defines a functor

$\mathrm{Fin}' \longrightarrow C$, $I \longmapsto M^I$.

$i \in I \in \mathrm{Fin}$, $\rho_i : I \supset \{i\} \longrightarrow *$ "Segal maps"

$\rightsquigarrow M^I \xrightarrow{\pi_i} M$

Prop Let C be a category with finite products. There is a fully faithful functor

$$\mathrm{CMon}(C) \hookrightarrow \mathrm{Fun}(\mathrm{Fin}', C)$$
$$M \longmapsto (I \longmapsto M^I)$$

A functor $X : \mathrm{Fin}' \longrightarrow C$ is in the essential image iff for every $I \in \mathrm{Fin}$,

$$X(I) \xrightarrow{(\rho_i)_{i \in I}} X(*)^I \qquad \text{"Segal condition"}$$

is an isomorphism.

Pf. Exercise.

Remark One can show that there is an equivalence of 2-categories

$$\mathrm{SymMonCat} \simeq \mathrm{Fun}_{\mathrm{Segal}}(\mathrm{Fin}', \mathrm{Cat}) \subset \mathrm{Fun}(\mathrm{Fin}', \mathrm{Cat})$$

full subcategory on $X : \mathrm{Fin}' \longrightarrow \mathrm{Cat}$ s.t.: $X(I) \xrightarrow[(\rho_i)]{} X(*)^I$ is an equivalence.

General monoids $M \in \mathrm{Mon}(C)$

To define $\mu_I : M^I \longrightarrow M$, we need an ordering on I.

Let Ord be the category of finite, totally ordered sets.

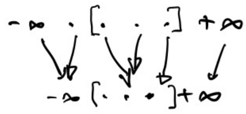
As before, M defines $\mu : \mathrm{Ord} \longrightarrow C$

$$I \longmapsto M^I$$

$$(J \xrightarrow{f} I) \longmapsto \mu_f : \begin{tikzcd} M^J \arrow[r] \arrow[d] & M^I \arrow[d, "\pi_i"] \\ M^{f^{-1}(i)} \arrow[r, "\mu"] & M \end{tikzcd}$$

Let $\mathrm{Ord}_{\pm\infty}$ be the subcategory of Ord with objects $I_{\pm\infty} = I \cup \{\pm\infty\}$
and morphisms satisfy $f(-\infty) = -\infty$
$f(+\infty) = +\infty$.

$$\begin{array}{ccc} \mathrm{Ord} \hookrightarrow \mathrm{Ord}_{\pm\infty} \\ I \longmapsto I_{\pm\infty} \end{array} \qquad \left(\text{analogous to: } \begin{array}{c} \mathrm{Fin} \hookrightarrow \mathrm{Fin}_* \\ I \longmapsto I_+ \end{array} \right)$$



Segal maps : $i \in I \in \mathrm{Ord}$, $\rho_i : I_{\pm\infty} \longrightarrow *_{\pm\infty}$

$$j \longmapsto \begin{cases} * & \text{if } j = i \\ -\infty & \text{if } j < i \\ +\infty & \text{if } j > i \end{cases}$$

<u>Prop</u> Let C be a category with finite products. There is a fully faithful functor
$$\mathrm{Mon}(C) \hookrightarrow \mathrm{Fun}(\mathrm{Ord}_{\pm\infty}, C)$$

A functor $X : \mathrm{Ord}_{\pm\infty} \to C$ is in the essential image iff for every $I \in \mathrm{Ord}$, $X(I_{\pm\infty}) \xrightarrow{(\rho_i)_{i \in I}} X(*_{\pm\infty})^I$ is an isomorphism

<u>Remark</u> One can show that there is an equivalence of 2-categories
$$\mathrm{MonCat} \simeq \mathrm{Fun}_{\mathrm{Segal}}(\mathrm{Ord}_{\pm\infty}, \mathrm{Cat}).$$

<u>Exercise</u> there is an equivalence $\mathrm{Ord}_{\pm\infty} \simeq \Delta^{op}$
$$\{1, \dots, n\}_{\pm\infty} \mapsto [n]$$
Under this equivalence, the Segal maps are $\rho_i : [1] \to [n]$ $(1 \leq i \leq n)$
$$0 \mapsto i-1$$
$$1 \mapsto i$$

<u>Remark</u> 1) The inclusion $\mathrm{CMon}(C) \hookrightarrow \mathrm{Mon}(C)$ is induced by the functor
$$\mathrm{Ord}_{\pm\infty} \longrightarrow \mathrm{Fin}_* \quad \text{sending } I_{\pm\infty} \text{ to } I_+.$$

2) For $C = \mathrm{Set}$, this is the nerve construction:
$$\mathrm{Mon} \hookrightarrow \mathrm{Cat} \xhookrightarrow{N} \mathrm{sSet}$$
$$M \longmapsto \left(* \leftleftarrows M \leftleftarrows M \times M \cdots \right)$$