Cellular homology

If X has a CW structure, define

$$\begin{tikzcd} C_*^{cell}(X) = \Big(\bigoplus\limits_{I_0} \mathbb{Z} & \cdots \arrow[l] & \bigoplus\limits_{\beta \in I_n} \mathbb{Z} \arrow[l] \arrow[d] & \bigoplus\limits_{\alpha \in I_{n+1}} \mathbb{Z} \arrow[l, "\partial"'] & \cdots \Big) \arrow[l] \\ & & \mathbb{Z}\beta & \mathbb{Z}\alpha \arrow[l, "\partial_{\alpha,\beta}"'] \arrow[u] \end{tikzcd}$$

$\alpha \in I_{n+1}$
$\beta \in I_n$.

$$\begin{tikzcd} D^n \arrow[r, "e_\beta"] & X^{(n)} \\ (D^n)^\circ \arrow[u] \arrow[ur, "\text{subspace}"'] \end{tikzcd}$$

$$S^n \xrightarrow{\partial e_\alpha} X^{(n)} \xrightarrow{\pi_\beta} S^n$$
collapse the complement of $(D^n)^\circ$ to a point

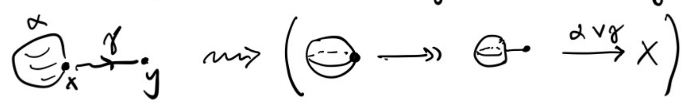
$\partial_{\alpha,\beta} : \mathbb{Z}\alpha \to \mathbb{Z}\beta$ is multiplication by the degree of $\pi_\beta \circ \partial e_\alpha$.

Thm $H_* C_*^{cell}(X) = H_* C_*^{sing}(X)$

Homology with coefficients in a local system.

Def Let X be a top. space and $\mathbf{C}$ a category. A local system on X with values in $\mathbf{C}$ is a functor $\Pi_1 X \longrightarrow \mathbf{C}$, where $\Pi_1 X$ is the fundamental groupoid of X.

Example $x \longmapsto \pi_n(X,x)$ is a local system of (abelian) groups on X $(n \geq 1)$



If $A : \Pi_1 X \longrightarrow Ab$, we can define $H_*(X, A)$ as follows.

Suppose X path connected and choose $x \in X$. Assume $\exists$ universal cover $(\tilde{X}, \tilde{x}) \longrightarrow (X, x)$.

$$\mathrm{Fun}(\Pi_1 X, Ab) \simeq \mathbb{Z}[\pi_1(X,x)]\text{-modules.}$$

$\pi_1(X,x)$ acts on $\tilde{X}$ hence it acts on $C_*^{sing}(\tilde{X})$

Def $C_*^{sing}(X, A) := C_*^{sing}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x)]} A$

$$H_*(X, A) = H_* C_*^{sing}(X, A).$$

If X has a CW structure, then $\tilde{X}$ has an induced CW structure s.t. the action of $\pi_1(X,x)$ permutes the cells.

$\Rightarrow \pi_1(X,x)$ acts on $C_*^{cell}(\tilde{X})$.

Let $C_*^{cell}(X,A) := C_*^{cell}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x)]} A$.

$\Rightarrow H_* C_*^{cell}(X,A) = H_*(X,A)$.

Example. $\pi_1(\mathbb{RP}^2) = C_2$. Let $\tilde{\mathbb{Z}}$ be the sign representation of C_2

$$H_*(\mathbb{RP}^2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & *=1 \\ 0 & *\geq 2 \end{cases} \qquad H_*(\mathbb{RP}^2,\tilde{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & *=0 \\ 0 & *=1 \\ \mathbb{Z} & *=2 \\ 0 & *\geq 3 \end{cases}$$

not orientable

Whitehead's theorem

A morphism $f: X \to Y$ in Top is a weak equivalence iff:

1) f induces an equivalence on Π_1. ($\Leftrightarrow$ iso on π_0 and π_1)

2) for every $A: \Pi_1 Y \to Ab$, $f_*: H_*(X, f^*A) \to H_*(Y,A)$ is iso.

$f^*A: \Pi_1 X \xrightarrow{\Pi_1 f} \Pi_1 Y \xrightarrow{A} Ab$

Def. A space X is simple (resp. nilpotent) if $\forall x \in X$, $\pi_1(X,x)$ is abelian (resp. nilpotent) and acts trivially (resp. nilpotently) on $\pi_n(X,x)$ for $n \geq 2$.

Examples • Simply connected $\Rightarrow$ simple $\Rightarrow$ nilpotent.

• If (X,x) is a pointed space, $\Omega_x X$ is simple.

• More generally, any connected H-space is simple.

(Eckmann-Hilton.)

[An H-space is a space X with $\mu: X \times X \to X$ and $e \in X$ s.t. $\mu(e,-)$ and $\mu(-,e) : X \to X$ are homotopic to the identity].

Theorem If X and Y are nilpotent, a map $f: X \to Y$ is a weak equivalence iff $f_*: H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$ is iso.