

Exercise sheet 8

Exercise 1. Let \mathcal{C} be an exact category. Show that there is an equivalence of categories $Q\mathcal{C} \simeq Q(\mathcal{C}^{\text{op}})$.

Exercise 2. Let X be a scheme.

- (a) Show that the category $\text{Vect}(X)$ of finite locally free \mathcal{O}_X -modules is an exact category if one takes admissible epimorphisms to be epimorphisms of sheaves.
- (b) Show that a morphism $u: \mathcal{E} \rightarrow \mathcal{F}$ in $\text{Vect}(X)$ is an admissible monomorphism if and only if, for every $x \in X$ with residue field $\kappa(x)$, the induced morphism of $\kappa(x)$ -vector spaces $u(x): \mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x) \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \kappa(x)$ is injective.

Exercise 3. Let \mathcal{C} be a small exact category. Show that there exists an abelian category \mathcal{A} and a fully faithful functor $h: \mathcal{C} \hookrightarrow \mathcal{A}$ such that:

- (a) the essential image of h is closed under extensions: if

$$0 \rightarrow h(A) \rightarrow X \rightarrow h(C) \rightarrow 0$$

is a short exact sequence in \mathcal{A} , then $X \simeq h(B)$ for some $B \in \mathcal{C}$;

- (b) a sequence in \mathcal{C} is exact if and only if its image by h is a short exact sequence in \mathcal{A} .

Hint. Let $\mathcal{A} \subset \text{Fun}(\mathcal{C}^{\text{op}}, \text{Ab})$ be the full subcategory of functors that are additive and send short exact sequences to left exact sequences.

Exercise 4. A ring R is called *flasque* if there exists an R -bimodule M , finitely generated and projective as a left R -module, and an isomorphism of bimodules $\theta: R \oplus M \simeq M$. An example is the ring of A -linear endomorphisms of $\bigoplus_{\mathbb{N}} A$ for any ring A .

Let R be a flasque ring.

- (a) Show that $K_0(R) = 0$.
- (b) More generally, show that the space $K(R)$ is weakly contractible as follows. Construct a finite-product-preserving functor $\infty: \text{Proj}(R) \rightarrow \text{Proj}(R)$ with a natural equivalence $\infty(P) \simeq P \oplus \infty(P)$ (here, $\text{Proj}(R)$ is the category of finitely generated projective *left* R -modules). Using the universal property of group completion, deduce that there exists a morphism $\infty: K(R) \rightarrow K(R)$ which is homotopic to $\text{id} + \infty$, where $+: K(R) \times K(R) \rightarrow K(R)$ is induced by the functor $\oplus: \text{Proj}(R) \times \text{Proj}(R) \rightarrow \text{Proj}(R)$. Using Eckmann–Hilton, deduce that $\pi_n K(R) = 0$ for all $n \geq 1$.

Remark. It follows from the group completion theorem that the space $BGL(R)$ is acyclic.