Exercise sheet 7

Exercise 1. Let G and H be groups. Two morphisms $\phi, \psi\colon G \to H$ are called *conjugate* if there exists $h \in H$ such that $\phi(g) = h^{-1}\psi(g)h$ for all $g \in G$. Show that in this case ϕ and ψ induce homotopic maps between classifying spaces $BG \to BH$ (in particular, they induce the same map on homology).

Exercise 2. Let $f\colon X \to Y$ be an acyclic morphism. Prove the following statements:

(a) For every $x \in X$, the map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective and has perfect kernel.

(b) If the fundamental groups of X have no nontrivial perfect subgroups, then f is a weak equivalence.

Exercise 3.

(a) Show that $(X \times Y)^+ \simeq X^+ \times Y^+$ for every CW complexes X and Y.

(b) Deduce that $K(R \times S) \simeq K(R) \times K(S)$ for every rings R and S.

Exercise 4. Let $\mathcal{C}$ be a category with finite products. Then $\mathcal{C}$ admits a canonical symmetric monoidal structure in which the monoidal product is the categorical product. In this exercise, we will see how to directly obtain the corresponding "unbiased" symmetric monoidal structure $\mathrm{Fin}' \to \mathrm{Cat}$, where Fin' is the category of finite sets and partially defined maps. Proceed in the following steps:

(a) There is a functor $\mathrm{Fin}' \to \mathrm{Cat}^{\mathrm{op}}$ sending a finite set I to its poset of subsets $\mathrm{Sub}(I)$. Hence there is a functor $\mathrm{Fin}' \to \mathrm{Cat}$ sending I to $\mathrm{Fun}(\mathrm{Sub}(I)^{\mathrm{op}}, \mathcal{C})$.

(b) Let $\mathcal{C}(I) \subset \mathrm{Fun}(\mathrm{Sub}(I)^{\mathrm{op}}, \mathcal{C})$ be the full subcategory of functors $X\colon \mathrm{Sub}(I)^{\mathrm{op}} \to \mathcal{C}$ such that, for every $J \subset I$, the map $X(J) \to \prod_{i \in J} X(\{i\})$ is an isomorphism. Then $I \mapsto \mathcal{C}(I)$ defines a subfunctor of $I \mapsto \mathrm{Fun}(\mathrm{Sub}(I)^{\mathrm{op}}, \mathcal{C})$.

(c) The functor $\mathrm{Fun}(\mathrm{Sub}(I)^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}^I$, $X \mapsto (X(\{i\}))_{i \in I}$, restricts to an equivalence of categories $\mathcal{C}(I) \simeq \mathcal{C}^I$ (*Hint:* $\mathcal{C}(I)$ is the essential image of the right adjoint). Hence, the functor $\mathcal{C}(-)\colon \mathrm{Fin}' \to \mathrm{Cat}$ satisfies the Segal condition.