Exercise sheet 6

Exercise 1. Let Ord be the category of finite ordered sets. Let $\mathrm{Ord}_{\pm\infty} \subset \mathrm{Ord}$ be the subcategory whose objects are of the form $I_{\pm\infty} = I \cup \{\pm\infty\}$ and whose morphisms satisfy $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. Given $I \in \mathrm{Ord}$ and $i \in I$, the *Segal map* ρ_i is

$$\rho_i \colon I_{\pm\infty} \to \{*\}_{\pm\infty}, \quad \rho_i(j) = \begin{cases} * & \text{if } j = i, \\ -\infty & \text{if } j < i, \\ +\infty & \text{if } j > i. \end{cases}$$

Show that there is an equivalence of categories $\mathrm{Ord}_{\pm\infty} \simeq \Delta^{\mathrm{op}}$ sending $\{1, \dots, n\}_{\pm\infty}$ to $[n]$ and the Segal map ρ_i, for $1 \leq i \leq n$, to (the opposite of) the morphism $[1] \to [n]$, $0 \mapsto i-1$, $1 \mapsto i$.

Exercise 2. Let $\mathcal{C}$ be a category with finite products.

(a) Show that there is a fully faithful functor

$$\mathrm{Mon}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

whose essential image consists of the simplicial objects X for which the Segal maps $\rho_i \colon [1] \to [n]$ induce isomorphisms $X([n]) \simeq X([1])^n$ for all $[n] \in \Delta$.

Remark. More generally, one can identify category objects in $\mathcal{C}$ with simplicial objects $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that the Segal maps induce isomorphisms between $X([n])$ and the n-fold fiber products $X([1]) \times_{X([0])} X([1]) \times_{X([0])} \cdots \times_{X([0])} X([1])$.

(b) Show that there is a fully faithful functor

$$\mathrm{CMon}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$$

whose essential image consists of the functors X for which the Segal maps $\rho_i \colon I_+ \to *_+$ induce isomorphisms $X(I_+) \simeq X(*_+)^I$ for all $I \in \mathrm{Fin}$.

Exercise 3. Let $\tilde{\mathbb{Z}}$ be the sign representation of $C_2 = \pi_1(\mathbb{RP}^2)$. Use cellular homology to compute $H_*(\mathbb{RP}^2, \tilde{\mathbb{Z}})$.

Exercise 4. Recall that a group P is *perfect* if $P = [P, P]$. Show that every group G has a maximal perfect subgroup P, which can be described in the following two ways:

(a) P is the union of all perfect subgroups of G;

(b) P is the limit of the transfinite derived series $G^{(\alpha)}$ of G, defined by:

$$G^{(\alpha)} = \begin{cases} G & \text{if } \alpha = 0, \\ [G^{(\beta)}, G^{(\beta)}] & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta < \alpha} G^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$