# Exercise sheet 5

**Exercise 1.** Let $R$ be a ring and $n \geq 1$. Let $M_n(R)$ be the ring of $n \times n$ matrices with coefficients in $R$. Prove that the categories $\mathrm{Proj}(R)$ and $\mathrm{Proj}(M_n(R))$ of finitely generated projective left modules are equivalent, hence that $K_i(R) \simeq K_i(M_n(R))$ for $i = 0, 1$.

**Exercise 2.** An object $X$ in a category has the *(unique) right lifting property* with respect to a morphism $Y \to Z$ if every morphism $Y \to X$ factors (uniquely) through $Z$.

Prove the following statements:

(a) The nerve functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful.

(b) A simplicial set $X$ is isomorphic to $N(C)$ for some category $C$ if and only if $X$ has the unique right lifting property with respect to the inclusions $\Lambda^n_i \subset \Delta^n$ for $0 < i < n$.

(c) A simplicial set $X$ is isomorphic to $N(C)$ for some groupoid $C$ if and only if $X$ has the unique right lifting property with respect to the inclusions $\Lambda^n_i \subset \Delta^n$ for $0 \leq i \leq n$.

**Exercise 3.** Recall that a simplicial set is a *Kan complex* if it has the right lifting property with respect to the inclusions $\Lambda^n_i \subset \Delta^n$ for $0 \leq i \leq n$. Show that every simplicial group is a Kan complex.

**Exercise 4.** Let $p\colon E \to B$ be a functor between groupoids and let $e_0 \in E$, $b_0 = p(e_0)$. The *homotopy fiber* $F$ of $p$ at $b_0$ is the groupoid whose objects are pairs $(e, \gamma)$ with $e \in E$ and $\gamma\colon p(e) \simeq b_0$ and with the obvious morphisms. Let $i\colon F \to E$ be the functor $(e, \gamma) \mapsto e$ and let $f_0 = (e_0, \mathrm{id}) \in F$.

(a) Construct an action of $\pi_1(B, b_0)$ on $\pi_0(F)$ and a sequence

$$\pi_1(F, f_0) \xrightarrow{i_*} \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B)$$

which is exact in the following sense:

(1) exactness at $\pi_1(E, e_0)$: $\ker(p_*) = \mathrm{im}(i_*)$

(2) exactness at $\pi_1(B, b_0)$: $\partial\alpha = \partial\beta$ if and only if $\alpha^{-1}\beta \in \mathrm{im}(p_*)$

(3) exactness at $\pi_0(F)$: $\partial$ is $\pi_1(B, b_0)$-equivariant and $i_*(a) = i_*(b)$ if and only of $a$ and $b$ are in the same orbit

(4) exactness at $\pi_0(E)$: $(p_*)^{-1}([b_0]) = \mathrm{im}(i_*)$

(b) Let $p\colon E \to B$ be a continuous map between compactly generated topological spaces[1] and let $e_0 \in E$, $b_0 = p(e_0)$. The *homotopy fiber* $F$ of $p$ at $b_0$ is the set of pairs $(e, \gamma)$ with $e \in E$ and $\gamma$ a path from $p(e)$ to $b_0$ in $B$, topologized as a subspace of $E \times \mathrm{Hom}([0, 1], B)$. Without going into the details, explain how (a) induces a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F, f_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, f_0) \to \cdots .$$

[1] The definition of compactly generated topological space is not very important here; what matters is that they satisfy the *exponential law* $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$, where $\mathrm{Hom}(A, B)$ is the space of continuous maps with the compact-open topology.