Exercise sheet 4

Exercise 1. Let R be a ring and $I \subset R$ a two-sided nilpotent ideal. Show that $K_1(R) \to K_1(R/I)$ is surjective. Give an example showing that it is not injective in general.

Exercise 2.

(a) Let R be a noetherian commutative ring of Krull dimension d. Use the Bass–Serre cancellation theorem to show that the map

$$\mathbb{Z}_R \times \pi_0(\mathrm{Proj}_n(R)^{\simeq}) \to K_0(R), \quad (r, P) \mapsto [R^r] + [P] - n,$$

is surjective if $n \geq d$ and injective if $n > d$.

(b) Let R be an arbitrary commutative ring and let $x \in K_0(R)$ be of rank ≥ 1. Show that there exists $m \geq 0$ and $P \in \mathrm{Proj}(R)$ such that $mx = [P]$.

Hint. If R is noetherian of finite Krull dimension, use (a). In general, write R as a filtered colimit of rings of the form $\mathbb{Z}[x_1, \ldots, x_n]/I$, which are noetherian of finite Krull dimension.

Exercise 3. Let $f\colon R \to S$ be a morphism of commutative rings such that S is finitely generated and projective as an R-module. In this case, the extension of scalars functor $f^*\colon \mathrm{Proj}(R) \to \mathrm{Proj}(S)$ has a right adjoint $f_*\colon \mathrm{Proj}(S) \to \mathrm{Proj}(R)$, and there are induced morphisms $f^*\colon K_0(R) \to K_0(S)$ and $f_*\colon K_0(S) \to K_0(R)$.

(a) Show that the *projection formula* holds:

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

for all $x \in K_0(S)$ and $y \in K_0(R)$ (in other words, f_* is a morphism of $K_0(R)$-modules).

(b) Describe the composites $f_* f^*\colon K_0(R) \to K_0(R)$ and $f^* f_*\colon K_0(S) \to K_0(S)$.

(c) If S is free of rank n as an R-module, deduce that $K_0(R)[1/n] \simeq K_0(S)[1/n]$.

Exercise 4. Let R be a commutative ring. Construct a $K_0(R)$-module structure on $K_1(R)$ such that, for $P \in \mathrm{Proj}(R)$ and $\phi \in \mathrm{Aut}_R(Q)$, $[P] \cdot [\phi] = [\mathrm{id}_P \otimes \phi]$.