Exercise sheet 3

Exercise 1. Let $\mathcal{A}$ be a category with finite products. Let $\mathrm{Mon}(\mathcal{A})$ be the category of monoids in $\mathcal{A}$ and $\mathrm{CMon}(\mathcal{A}) \subset \mathrm{Mon}(\mathcal{A})$ the subcategory of commutative monoids. Show that there are equivalences of categories:

(a) $\mathrm{CMon}(\mathcal{A}) \simeq \mathrm{Mon}(\mathrm{Mon}(\mathcal{A}))$

(b) $\mathrm{Mon}(\mathrm{CMon}(\mathcal{A})) \simeq \mathrm{CMon}(\mathcal{A})$

Exercise 2. Let M be a commutative monoid and $L \subset M$ a cofinal submonoid (that is, for every $x \in M$, there exists $y \in M$ such that $x + y \in L$). Prove the following assertions:

(a) The induced moprhism $L^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is injective.

(b) The map $M \times L \to M^{\mathrm{gp}}$, $(x, y) \mapsto x - y$, exhibits M^{gp} as the quotient of $M \times L$ by the equivalence relation:

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad \text{there exists } z \in L \text{ such that } x + y' + z = x' + y + z.$$

Remark. In the special case $L = \mathbb{N}$, we can reformulate this description as M^{gp} as follows:

$$M^{\mathrm{gp}} \simeq \operatorname{colim}\left(M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \cdots\right).$$

Exercise 3. Let R be commutative ring. An *orientation* of a finitely generated projective R-module P is an isomorphism $\omega\colon \det(P) \simeq R$. Such pairs (P, ω) form a groupoid in which a morphism $(P, \omega) \to (P', \omega')$ is an isomorphism $\phi\colon P \to P'$ such that $\omega = \omega' \circ \det(\phi)$. Define $(P, \omega) \oplus (P', \omega')$ to be the R-module $P \oplus P'$ with orientation

$$\det(P \oplus P') \simeq \det(P) \otimes_R \det(P') \xrightarrow{\omega \otimes \omega'} R \otimes_R R \simeq R.$$

It is straightforward to check that the operation $\oplus$ is part of a symmetric monoidal structure on the groupoid $\mathrm{Proj}^{\mathrm{or}}_{\mathrm{ev}}(R)$ of oriented finitely generated projective R-modules of even rank.[1] The rank defines a morphism of abelian groups $\mathrm{rk}\colon \pi_0(\mathrm{Proj}^{\mathrm{or}}_{\mathrm{ev}}(R))^{\mathrm{gp}} \to 2\mathbb{Z}_R$. Let $\tau_{\leq 1} SK(R) \subset \mathrm{Proj}^{\mathrm{or}}_{\mathrm{ev}}(R)^{\mathrm{gp}}$ be the full subgroupoid consisting of the objects of rank 0; it is a Picard groupoid. Use the group completion theorem to show that

$$\pi_0(\tau_{\leq 1} SK(R)) \simeq SK_0(R) \quad \text{and} \quad \pi_1(\tau_{\leq 1} SK(R)) \simeq SK_1(R).$$

[1] The restriction to even rank is needed for the braiding to be orientation-preserving; without this restriction, $\oplus$ is only a non-braided monoidal structure.