Exercise sheet 2

Exercise 1. Let M and N be commutative monoids and $f\colon M \to N$ an arbitrary map. We say that f is polynomial of degree ≤ -1 if $f = 0$. For $n \geq 0$, we say that f is polynomial of degree $\leq n$ if for every $x \in M$ there exists a map $D_x(f)\colon M \to N$, polynomial of degree $\leq n-1$, such that

$$f(y+x) = f(y) + D_x(f)(y)$$

for all $y \in M$. Denote by

$$\mathrm{Poly}_{\leq n}(M,N)$$

the set of polynomial maps $M \to N$ of degree $\leq n$.

Consider the monoid ring $\mathbb{Z}[M]$. For $x \in M$, denote by $[x]$ the corresponding element in $\mathbb{Z}[M]$, so that $[x+y] = [x][y]$. Let $\epsilon\colon \mathbb{Z}[M] \to \mathbb{Z}$ be the augmentation map sending every $[x]$ to 1, let $I = \ker(\epsilon)$ be the augmentation ideal, and let I^n be the nth power of the ideal I. Given an abelian group A and a map $f\colon M \to A$, write $\hat{f}\colon \mathbb{Z}[M] \to A$ for the unique additive extension of f.

(a) Show that $f\colon M \to A$ is polynomial of degree $\leq n$ if and only if $\hat{f}(I^{n+1}) = 0$. Deduce that $\mathrm{Poly}_{\leq n}(M,A) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[M]/I^{n+1}, A)$.

(b) Show that the morphism of rings $\mathbb{Z}[M]/I^n \to \mathbb{Z}[M^{\mathrm{gp}}]/I^n$ induced by the canonical map $M \to M^{\mathrm{gp}}$ is an isomorphism (*Hint:* observe that $\mathbb{Z}[M^{\mathrm{gp}}] \simeq M^{-1}\mathbb{Z}[M]$). Deduce that every polynomial map $f\colon M \to N$ extends uniquely to a polynomial map $M^{\mathrm{gp}} \to N^{\mathrm{gp}}$.

Exercise 2. Let R be a commutative ring and let $P_n\colon \mathrm{Proj}(R) \to \mathrm{Proj}(R)$, $n \geq 0$, be a family of functors with natural isomorphisms

$$P_0(M) \simeq R,$$
$$P_n(M \oplus N) \simeq \bigoplus_{i+j=n} P_i(M) \otimes_R P_j(N).$$

Examples include: $(-)^{\otimes n}$, Sym^n, Γ^n, Λ^n.

Show that the map $p_n\colon \pi_0(\mathrm{Proj}(R)^{\simeq}) \to \pi_0(\mathrm{Proj}(R)^{\simeq})$ defined by $p_n([M]) = [P_n(M)]$ is polynomial of degree $\leq n$ and hence extends uniquely to a polynomial map $p_n\colon K_0(R) \to K_0(R)$.

Remark. The maps so obtained from the exterior powers are denoted by $\lambda^n\colon K_0(R) \to K_0(R)$. They give the name to the notion of λ-*ring* which is a commutative ring equipped with self-maps λ^n satisfying some identities; $K_0(R)$ is thus an example of a λ-ring.

Exercise 3. Let R be a commutative ring. Denote by $\mathbb{Z}_R$ the ring of continuous maps $\mathrm{Maps}(\mathrm{Spec}\, R, \mathbb{Z})$. Recall that $SK_0(R) \subset K_0(R)$ is the kernel of the map

$$(\mathrm{rk}, \det)\colon K_0(R) \to \mathbb{Z}_R \times \mathrm{Pic}(R).$$

(a) Construct a $\mathbb{Z}_R$-module structure on the abelian group $\mathrm{Pic}(R)$.

(b) Show that the map $(\mathrm{rk}, \det)$ is a morphism of rings if we regard $\mathbb{Z}_R \times \mathrm{Pic}(R)$ as the square zero extension of $\mathbb{Z}_R$ by $\mathrm{Pic}(R)$ (i.e., with the ring structure given by $(a, x)(b, y) = (ab, ay + bx)$).

Hence, $SK_0(R)$ is an ideal in $K_0(R)$.

Exercise 4. Let R be a ring. Recall that $E_n(R) \subset GL_n(R)$ is the subgroup generated by the elementary matrices $e_{ij}(r)$.

(a) Show that $E_n(R)$ is perfect for $n \geq 3$, i.e., $E_n(R) = [E_n(R), E_n(R)]$.

Hint. Use the easily checked formula

$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l, \\ e_{il}(rs) & \text{if } j = k \text{ and } i \neq l, \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

(b) Let $g, h \in GL_n(R)$. Show that $[g, h] \oplus 1_n \in GL_{2n}(R)$ belongs to $E_{2n}(R)$.

Hint. Use the identities

$$\begin{pmatrix} [g,h] & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}$$
$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and show that every triangular matrix in $GL_n(R)$ with 1's on the diagonal belongs to $E_n(R)$.

(c) Deduce from (a) and (b) that $E(R) = [GL(R), GL(R)]$.