Exercise sheet 12

Exercise 1. Let D be a Dedekind domain with fraction field F. Show that there is a long exact sequence

$$\cdots \to K_{i+1}(F) \to \bigoplus_{\mathfrak{m}\subset D} K_i(D/\mathfrak{m}) \to K_i(D) \to K_i(F) \to \cdots \to K_0(F) \to 0,$$

where $\mathfrak{m}$ ranges over the maximal ideals in D.

Exercise 2. Let X be a noetherian scheme, union of two open subschemes U_1 and U_2. Show that there is a long exact sequence

$$\cdots \to G_{i+1}(U_1\cap U_2) \to G_i(X) \to G_i(U_1)\oplus G_i(U_2) \to G_i(U_1\cap U_2) \to \cdots \to G_0(U_1\cap U_2) \to 0.$$

Exercise 3. Let X be a noetherian scheme and $\pi\colon X' \to X$ a finite morphism. Let $U \subset X$ be an open subset such that $U' = \pi^{-1}(U) \to U$ is an isomorphism. Let $Z = X - U$ be the closed complement (with any scheme structure) and let $Z' = \pi^{-1}(Z)$.

Since π is both affine and proper, one knows that the pushforward functor

$$\pi_*\colon \mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$$

is exact and preserves coherence. Hence, it induces a morphism $\pi_*\colon G(X') \to G(X)$.

(a) Show that there is a long exact sequence

$$\cdots \to G_{i+1}(X) \to G_i(Z') \to G_i(Z) \oplus G_i(X') \to G_i(X) \to \cdots \to G_0(X).$$

(b) Let k be a noetherian commutative ring with $2 \in k^\times$ and let

$$X = \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2))$$

be the affine nodal cubic over k. Use (a) to show that $G_n(X) \simeq G_n(k) \oplus G_{n-1}(k)$ for $n \geq 1$.

Hint. The map $\pi\colon \mathbb{A}^1_k \to X$ with $\pi^*(x) = t^2 - 1$ and $\pi^*(y) = t^3 - t$ is finite and is an isomorphism over the complement of the node. Applying (a) to this map shows that $G_n(X)$ is an extension of $G_{n-1}(k)$ by $G_n(\mathbb{A}^1_k) \simeq G_n(k)$. To find a splitting of this extension, consider a point of X which is not the node, e.g., $(-1, 0)$.

(c) Let k be a noetherian commutative ring of characteristic 2 and let X be as in (b). Compute $G_n(X)$ for $n \geq 1$.

Remark. One can also show by hand that the last map $G_0(Z) \oplus G_0(X') \to G_0(X)$ in (a) is surjective, hence that $G_0(X) \simeq G_0(k)$ in (b) and (c).