Exercise sheet 11

Exercise 1. Let $\mathcal{C}$ be an exact category and $\mathcal{P} \subset \mathcal{C}$ a full subcategory closed under extensions. Suppose that:

(1) for every exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, if $Y, Z \in \mathcal{P}$, then $X \in \mathcal{P}$;

(2) for every $X \in \mathcal{C}$, there exists an admissible epimorphism $P \twoheadrightarrow X$ with $P \in \mathcal{P}$.

Let $\mathcal{P}_n \subset \mathcal{C}$ be the full subcategory of objects having a $\mathcal{P}$-resolution of length $\leq n$. Prove the following statements for every $n \geq 0$:

(a) $\mathcal{P}_n$ is closed under extensions in $\mathcal{C}$.

(b) If $X \rightarrowtail Y \twoheadrightarrow Z$ is an exact sequence in $\mathcal{C}$ with $Y \in \mathcal{P}_n$ and $Z \in \mathcal{P}_{n+1}$, then $X \in \mathcal{P}_n$.

Exercise 2. A full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if it contains 0 and is closed under subobjects, quotients, and extensions. In this situation, $\mathcal{B}$ is an abelian category and the quotient $\mathcal{A}/\mathcal{B}$ exists in the 2-category of abelian categories and exact functors. The category $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$ and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \operatorname*{colim}_{X' \subset X, Y' \subset Y} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y'),$$

where the colimit is taken over all subobjects $X' \subset X$ and $Y' \subset Y$ such that $X/X' \in \mathcal{B}$ and $Y' \in \mathcal{B}$.

Let X be a noetherian scheme, $Z \subset X$ a closed subscheme, and $U \subset X$ the open complement of Z. Let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory of sheaves $\mathcal{F}$ such that $\mathcal{F}|_U = 0$. Show that $\mathrm{Coh}_Z(X)$ is a Serre subcategory of $\mathrm{Coh}(X)$ and that the restriction functor $\mathrm{Coh}(X) \to \mathrm{Coh}(U)$, $\mathcal{F} \mapsto \mathcal{F}|_U$, induces an equivalence of categories

$$\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \simeq \mathrm{Coh}(U).$$

Hint. The following standard fact is useful: every quasi-coherent sheaf on a noetherian scheme is the filtered union of its coherent subsheaves.

Exercise 3. Let $\mathcal{C}$ be an exact category. Show that there is a canonical morphism of E_∞-spaces

$$|N(\mathcal{C}^{\simeq})|^{\mathrm{gp}} \to K(\mathcal{C})$$

inducing the quotient map $\pi_0(\mathcal{C}^{\simeq})^{\mathrm{gp}} \twoheadrightarrow K_0(\mathcal{C})$ on π_0.

Hint. A map $T \to K(\mathcal{C})$ is the same thing as a self-homotopy of the constant map $T \to |N(Q\mathcal{C})|$, $t \mapsto 0$.