Exercise sheet 10

Exercise 1. Let $\mathcal{C}$ be an exact category and $\mathcal{B} \subset \mathcal{C}$ a full subcategory closed under extensions (with the induced exact structure). Suppose that, for every $X \in \mathcal{C}$, there exists $X' \in \mathcal{C}$ such that $X \oplus X' \in \mathcal{B}$ (one says that $\mathcal{B}$ is *cofinal* in $\mathcal{C}$). Show that the induced map $K_0(\mathcal{B}) \to K_0(\mathcal{C})$ is injective.

Hint. For the minimal exact structure on $\mathcal{C}$, this was proved in Exercise 3.2. Use this case as the starting point.

Remark. The *cofinality theorem* states that furthermore $K_n(\mathcal{B}) \simeq K_n(\mathcal{C})$ for all $n \geq 1$.

Exercise 2. Let $\mathcal{C}$ be an exact category. A chain complex (C_*, d_*) in $\mathcal{C}$ is called *exact* if each differential $d_{i+1} \colon C_{i+1} \to C_i$ factors as $C_{i+1} \twoheadrightarrow C'_i \rightarrowtail C_i$ and $C'_i \rightarrowtail C_i \twoheadrightarrow C'_{i-1}$ is an exact sequence.

Let $\mathcal{C}$ and $\mathcal{D}$ be exact categories and let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

be a chain complex of exact functors $\mathcal{C} \to \mathcal{D}$ which is objectwise exact in the above sense. Show that there is a null-homotopy

$$\sum_{i=0}^{n} (-1)^i (F_i)_* \simeq 0 \colon K(\mathcal{C}) \to K(\mathcal{D}).$$

Hint. To prove that the functors F'_i are exact, one can assume that $\mathcal{D}$ is abelian by Exercise 8.3.

Exercise 3. Let X be a noetherian scheme, $i \colon Z \hookrightarrow X$ a closed immersion, and $j \colon U \hookrightarrow X$ the complementary open immersion. Let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory of coherent sheaves $\mathcal{F}$ such that $j^*(\mathcal{F}) = 0$. Prove the following statements:

(a) The functor $i_* \colon \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$ is fully faithful and restricts to a functor $i_* \colon \mathrm{Coh}(Z) \to \mathrm{Coh}_Z(X)$.

(b) For every sheaf $\mathcal{F} \in \mathrm{Coh}_Z(X)$, there exists a finite filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is in the essential image of $i_* \colon \mathrm{Coh}(Z) \to \mathrm{Coh}_Z(X)$.

Exercise 4. Let k be a field. Recall the classification of vector bundles (finite locally free sheaves) on the projective line $\mathbb{P}^1_k$: every vector bundle is a sum of line bundles in a unique way, and every line bundle is isomorphic to $\mathcal{O}(n)$ for some unique $n \in \mathbb{Z}$. Using this result, compute and compare the following abelian groups:

(a) $\pi_0(\mathrm{Vect}(\mathbb{P}^1_k)^{\simeq})^{\mathrm{gp}}$

(b) $K_0(\mathrm{Vect}(\mathbb{P}^1_k))$

Hint. There is a short exact sequence of sheaves $0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0$.