

Exercise sheet 1

Exercise 1. Recall that the group completion functor $(-)^{\text{gp}}: \text{CMon} \rightarrow \text{Ab}$ is left adjoint to the inclusion $\text{Ab} \subset \text{CMon}$.

- (a) A commutative monoid M has the *cancellation property* if $a + c = b + c$ implies $a = b$. Show that the unit map $M \rightarrow M^{\text{gp}}$ is injective if and only if M has the cancellation property.
- (b) Suppose that $(M, +, \cdot)$ is a semiring (i.e., $(M, +)$ is a commutative monoid, (M, \cdot) is a monoid, and the distributivity law holds). Show that the group completion M^{gp} (with respect to $+$) admits a unique ring structure such that $M \rightarrow M^{\text{gp}}$ is a morphism of semirings.
- (c) Show that the inclusion $\text{Ab} \subset \text{CMon}$ also has a right adjoint and describe it explicitly.
- (d) Let X be a topological space. Show that $\text{Maps}(X, \mathbb{N})^{\text{gp}} \simeq \text{Maps}(X, \mathbb{Z})$, where $\text{Maps}(X, Y)$ denotes the set of continuous maps and \mathbb{N} and \mathbb{Z} are viewed as discrete topological spaces.
- (e) Let Mon be the category of monoids and Grp the category of groups. Show that the inclusion $\text{Grp} \subset \text{Mon}$ has a left adjoint L and that if M is a commutative monoid then $L(M) \simeq M^{\text{gp}}$.

Exercise 2.

- (a) Let G be a finite group. Show that the functor associating to a finite G -set X the G -representation $\mathbb{C}[X]$ induces a morphism of rings $A(G) \rightarrow R(G)$ from the Burnside ring to the representation ring (use Exercise 1(b)).
- (b) Let p be a prime number and C_p a cyclic group of order p . Show that there are isomorphisms of rings

$$A(C_p) \simeq \mathbb{Z}[x]/(x^2 - px) \quad \text{and} \quad R(C_p) \simeq \mathbb{Z}[y]/(y^p - 1).$$

What is the map $A(C_p) \rightarrow R(C_p)$ from (a) under these identifications?

Exercise 3. Let $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and let $P = \ker(\sigma)$ where

$$\sigma: A^3 \rightarrow A, \quad \sigma(f, g, h) = xf + yg + zh.$$

Show that there is an isomorphism of A -modules $P \oplus A \simeq A^3$ but that P is not free.

Hint. Let S^2 be the sphere of unit vectors in \mathbb{R}^3 . Observe that every $p \in P$ induces a continuous function $S^2 \rightarrow \mathbb{R}^3$. Use this to show that if P were free, then the tangent bundle to S^2 would be trivial, which is false.

Exercise 4. Let R be a ring and $I \subset R$ a two-sided nilpotent ideal (i.e., $I^n = 0$ for some n). Extension of scalars defines a functor $\text{Proj}(R) \rightarrow \text{Proj}(R/I)$, $P \mapsto P/IP$. Show that:

- (a) For every $P, Q \in \text{Proj}(R)$, the induced map $\text{Isom}_R(P, Q) \rightarrow \text{Isom}_{R/I}(P/IP, Q/IQ)$ is surjective, where Isom_R denotes the set of R -linear isomorphisms.
- (b) $\text{Proj}(R) \rightarrow \text{Proj}(R/I)$ induces a bijection between isomorphism classes of objects.

In particular, $K_0(R) \rightarrow K_0(R/I)$ is an isomorphism (we say that K_0 is *nilinvariant*).

Hint. For both (a) and (b), reduce first to the case $I^2 = 0$. Note that the injectivity in (b) follows from (a). To prove the surjectivity in (b), first show that every idempotent $\bar{e} \in R/I$ lifts to an idempotent in R . If e is any lift of \bar{e} , this amounts to solving the equation $(e+x)^2 = e+x$ in R ; this equation becomes easier to solve by adding the constraint $ex = xe$.