The partial K-theory of $\mathbb{F}_p$

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Our goal in this talk is to prove the following theorem

Theorem 1. *The projection map*

$$\mathrm{K}^{part}(\mathbb{F}_p) \to \pi_0 \mathrm{K}^{part}(\mathbb{F}_p) \cong \mathbb{Z}_{\geq 0}$$

is an isomorphism on mod p homology.

1 A general formula for partial K-theory

Let $\mathcal{C}$ be a Waldhausen ∞-category. Recall that the partial K-theory of $\mathcal{C}$ is defined as $\mathbb{L}S_\bullet\mathcal{C}^{\simeq}$, where

$$\mathbb{L} : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Space}) \to \mathrm{Mon}(\mathrm{Space})$$

is the left adjoint to the canonical inclusion $\mathrm{Mon}(\mathrm{Space}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Space})$ identifying E_1-monoids as those Segal spaces with contractible 0-space.

Our goal for this section is to give a formula for $\mathbb{L}$. For this it will be convenient reintepret the indexing category Δ^{op}

Remark 2. *Let $\mathrm{Ord}_\pm$ be the category of finite totally ordered sets with a distinct top and bottom and morphisms preserving top and bottom. We will write the generic element of $\mathrm{Ord}_\pm$ as*

$$(n) = \{\bot < 1 < \cdots < n < \top\}\,.$$

Note that $(0) = \{\bot < \top\}$. Then there is an equivalence

$$\Delta^{\mathrm{op}} \cong \mathrm{Ord}_\pm\ .$$

This is obtained by sending a finite totally ordered set A to the poset of subsets $B' \subseteq A$ that are downward closed, with bottom $\varnothing$ and top A. Therefore, under this equivalence $[n]$ is sent to (n) (if we write i for the downward closed subset $\{0 < \cdots < i-1\}$).

Let $\mathrm{Ord}_\pm^\times$ be the product completion of $\mathrm{Ord}_\pm$. Its objects are finite collections $\{(n_i)\}_{i\in I}$ of objects of $\mathrm{Ord}_\pm$ and a morphism $\{(n_i)\}_{i\in I} \to \{(m_j)\}_{j\in J}$ is a pair $(\varphi, \{f_j\}_{j\in J})$ where $\varphi : J \to I$ is a map of finite sets and $f_j : (n_{\varphi j}) \to (n_i)$ is a map in $\mathrm{Ord}_\pm$. By abstract nonsense we have

$$\mathrm{Fun}(\mathrm{Ord}_\pm, \mathrm{Space}) \cong \mathrm{Fun}^\times(\mathrm{Ord}_\pm^\times, \mathrm{Space})\,.$$

Construction 3. *We define $\mathcal{I}$ as the following category. Its objects are pairs (I, P) where I is a finite totally ordered set and $P = \{P_j\}_{j\in J}$ is a partition of I into intervals. A morphism $f : (I, P) \to (I', P')$ is a monotone function $f : I \to J$ such that it refines the partition (i.e. every interval P_i is the union of subsets of the form $f^{-1}Q_j$.*

We will write the general element (I, P) as

$$(d_1)\cdots(d_k)$$

where $I = \{(1,1) < (1,2) < \cdots < (1,d_1) < (1,2) < \cdots < (k,1) < \cdots < (k,d_k)\}$ and $P_j = \{(j,1) < \cdots < (j,d_k)\}$.

There is a morphism $\mathcal{I} \to \mathrm{Ord}_\pm^\times$ sending $(n_1)\cdots(n_k)$ to $\{(n_1),\ldots,(n_k)\}$ (be warned: the (n_i) on both sides are elements of different categories! We need to add a top and bottom).

Theorem 4. *If $X \in \mathrm{Fun}(\mathrm{Ord}_\pm, \mathrm{Space})$ the left adjoint $\mathbb{L}(X)$ is the monoid with underlying space the colimit*

$$\operatorname*{colim}_{(n_1)\cdots(n_k)\in\mathcal{I}} X_{n_1} \times \cdots \times X_{n_k}\,.$$

of the composite functor

$$\mathcal{I} \to \mathrm{Ord}_\pm^\times \xrightarrow{X^\times} \mathrm{Space}$$

Proof. There is a functor

$$\mathrm{Free} : \mathrm{Ord}_\pm^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Ord}_\pm, \mathrm{Space}) \xrightarrow{\mathbb{L}} \mathrm{Mon}(\mathrm{Space})$$

sending (n) to the free monoid generated by $\{1,\ldots,n\}$ (this is just a simple application of Yoneda). We will write $\mathcal{T}$ for the opposite of its essential image (this is the Lawvere theory of associative monoids). Therefore we have a functor $\mathrm{Free} : \mathrm{Ord}_\pm \to \mathcal{T}$

Lemma 5. *Precomposition with* Free *induces an equivalence*

$$\mathrm{Fun}^\times(\mathcal{T}, \mathrm{Space}) \to \mathrm{Mon}(\mathrm{Space})$$

and therefore the functor $\mathrm{Mon}(\mathrm{Space}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Space})$ *can be identified with the functor*

$$\mathrm{Fun}^\times(\mathcal{T}, \mathrm{Space}) \to \mathrm{Fun}^\times(\mathrm{Ord}_\pm^\times, \mathrm{Space})$$

induced by precomposition with the unique product-preserving functor

$$\mathrm{Free}^\times : \mathrm{Ord}_\pm^\times \to \mathcal{T}\,.$$

Proof. This is just an easy consequence of [1, Proposition 5.5.8.25] since Free(1) is a compact projective generator of Mon(Space). ◻

Now, by a proof analogous to [2, Lemma 2.18] left Kan extension along a product preserving functor of functors valued in a cartesian closed category preserves product preserving functors (the functor $(A \times_B B_{/x})(A \times_B B_{/y}) \to A \times_B B_{/x\times y}$ sending $(\phi a \to x, \phi a' \to y)$ to $\phi(a \times a') \to x \times y)$ has a left adjoint and so it is cofinal). Therefore the left adjoint $\mathbb{L}$ to

$$\mathrm{Fun}^{\times}(\mathcal{T}, \mathrm{Space}) \to \mathrm{Fun}^{\times}(\mathrm{Ord}_{\pm}^{\times}, \mathrm{Space})$$

is given by the left Kan extension. To conclude we just need to compute the value of this left Kan extension on the object Free(1). By general nonsense this value is given by

$$\operatorname*{colim}_{\{(n_i)\}_{i\in I} \in \mathrm{Ord}_{\pm}^{\times} \times_{\mathcal{T}} \mathcal{T}_{/\,\mathrm{Free}(1)}} X(n_1) \times \cdots \times X(n_k)$$

Now $\mathrm{Ord}_{\pm}^{\times} \times_{\mathcal{T}} \mathcal{T}_{/\,\mathrm{Free}(1)}$ is the category of pairs $(\{(n_i)\}_{i\in I}, x)$ where $\{(n_i)\}_{i\in I} \in \mathrm{Ord}_{\pm}^{\times}$ and $x \in \mathrm{Free}(n_1 + \cdots + n_k)$. We construct a map

$$\mathcal{I} \to \mathrm{Ord}_{\pm}^{\times} \times_{\mathcal{T}} \mathcal{T}_{/\,\mathrm{Free}(1)}$$

sending

$$(n_1)\cdots(n_k) \mapsto (\{(n_i)\}_{i=1\dots,k}, x_1^{(1)} \cdots x_{n_1}^{(1)} \cdots x_1^{(k)} \cdots x_{n_k}^{(k)}$$

This turns out to have a left adjoint, and so it is cofinal, thus proving the theorem. (The description of this left adjoint is messy so we'll not do it here, but it essentially sends a monomial to the totally ordered sets of its variables with the coarser partition in convex subsets in variables of the same type). ◻

2 Partial K-theory of split-exact categories

Now let us suppose $\mathcal{C}$ is a split exact category (for example the category of finite dimensional vector spaces), that is an additive category $\mathcal{C}$ together with the Waldhausen structure where the cofibrations are morphisms isomorphic to morphisms $X \to X \oplus Y$.

Remark 6. $\mathrm{K}_0^{part}(\mathcal{C})$ *is just the monoid* $\pi_0\mathcal{C}^{\simeq}$ *of isoclasses of* $\mathcal{C}$.

We want to write a simpler formula for $\mathrm{K}^{part}(\mathcal{C})$. The first observation is that we can describe $\pi_0 S_n \mathcal{C}^{\simeq}$.

Definition 7. *A filtered dimension is a sequence* $\mathbf{d} = \langle d_1, \dots, d_n\rangle$ *of elements* $d_i = [X_i] \in \mathrm{K}_0^{part}(\mathcal{C})$. *We think of it as specifying an object*

$$0 \rightarrowtail X_1 \rightarrowtail X_1 \oplus X_2 \rightarrowtail \dots \rightarrowtail X_1 \oplus \cdots X_n$$

of $S_n\mathcal{C}$. Since $\mathcal{C}$ is split exact, there is a bijection between filtered dimensions and $\pi_0 S_n\mathcal{C}^{\simeq}$. We say that $\mathbf{d}$ is of lenght n and dimension $\sum_i d_i$. We write $l(\mathbf{d}) = n$ and $|\mathbf{d}| = \sum_i d_i$.

A filtered dimension sequence is just an ordered sequence $D = (\mathbf{d}^{(1)}, \mathbf{d}^{(2)}, \ldots, \mathbf{d}^{(k)})$ of filtered dimensions. We write $l(D) = \sum_j l(\mathbf{d}^{(j)})$ and $|D| = \sum_j |\mathbf{d}^{(j)}|$.

If $\mathbf{d}$ is a filtered dimension sequence, we write $S(\mathbf{d})$ for the connected component of $S_{l(\mathbf{d})}(\mathcal{C})^{\simeq}$ corresponding to $\mathbf{d}$, so that

$$S_n\mathcal{C}^{\simeq} = \coprod_{l(\mathbf{d})=n} S(\mathbf{d})$$

Now let $\mathcal{F}$ be the Grothendieck construction on the functor $\mathcal{I} \to \mathrm{Set}$ given by

$$(n_1)\cdots(n_k) \mapsto \pi_0(S_{n_1}\mathcal{C} \times \cdots \times S_{n_k}\mathcal{C})^{\simeq}\,.$$

This is a 1-category. Its objects are filtered dimension sequences $D = (\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(k)})$ and morphisms are arrows $f : (n_1)\cdots(n_k) \to (m_1)\cdots(m_r)$ in $\mathcal{I}$ such that $\mathbf{d}_j = \sum_{fi=j} \mathbf{d}_i$. Then we can write

$$\mathrm{K}^{part}(\mathcal{C}) = \operatorname*{colim}_{(n_1)\cdots(n_k)\in\mathcal{I}} S_{n_1}\mathcal{C}^{\simeq} \times \cdots \times S_{n_k}\mathcal{C}^{\simeq} \cong \operatorname*{colim}_{D\in\mathbb{F}} S(D)\,.$$

Concretely an element of $\mathcal{F}$ is an element $(n_1)\cdots(n_k) \in \mathcal{I}$ whose elements are labelled by elements in $\mathrm{K}_0^{part}\mathcal{C}$. Its morphisms are morphisms $f : I \to I'$ in $\mathcal{I}$ such that every element in the target is labelled with the sum of the labels of the elements in its preimage.

We can simplify $\mathcal{F}$ further. Note that $\mathrm{K}_0^{part}\mathcal{C} = \pi_0\mathcal{C}^{\simeq}$ is a zerosumfree monoid, since the product of two sets is the one-point set iff both of them are: therefore if an element in the target of a morphism in $\mathbb{F}$ is labeled with 0, all its preimages must be labeled with 0 as well. Let us say that a filtered dimension $\mathbf{d} = (d_1, \ldots, d_n)$ is reduced if none of the d_i is 0 (this means that it corresponds to a nondegenerate simplex of $\pi_0 S_\bullet\mathcal{C}^{\simeq}$).

Lemma 8. *Let $\mathcal{F}^{red} \subseteq \mathcal{F}$ be the full subcategory spanned by the filtered dimension sequences for which all components are reduced. Then the inclusion is cofinal and $\mathbb{F}^{red}$ decomposes*

$$\mathcal{F}^{red} = \coprod_{d\in\mathrm{K}_0^{part}(\mathcal{C})} \mathcal{F}^{red}_d$$

where $\mathcal{F}^{red}$ is subcategory of reduced filtered dimension sequences of total dimension d. Moreover if $\mathrm{K}_0^{part}\mathcal{C}$ is cancellative, all $\mathcal{F}^{red}_d$ are posets (in fact you need less: all you need is that if $m, \in \mathrm{K}_0^{part}\mathcal{C}$ are such that $m + n = m$, then $n = 0$, which is true for all categories of projective modules over a ring).

Proof. First of all notice that the inclusion $\mathcal{F}^{red} \subseteq \mathcal{F}$ has a left adjoint, obtained by discarding all null elements of a filtered dimension sequence (this is well-defined since an element in the target can be labeled with 0 only if all of its preimages are 0), therefore the inclusion is cofinal. The decomposition is obvious from the fact that all maps in $\mathcal{F}$ preserve the total dimension.

Finally the statement about being a poset follows from the fact that in a cancellative zerosummandfree monoid M, for every list $\{m_1, m_2, \dots\}$ of elements in M and $n \in N$ there is at most one j such that $m_1 + \dots + m_j = n$. □

In $\mathcal{F}_n^{red}$ there are two special kinds of maps: the collapse maps, that are maps that preserve the partition in convex subsets, for example

$$c_i \langle d_1, \dots, d_n \rangle \mapsto \langle d_1, \dots, d_{i-1}, d_i + d_{i+1}, d_{i+2}, \dots d_n \rangle$$

and the splitting maps, that are maps that are bijections on the underlying sets and preserve the labels. For example

$$s_i : \langle d_1, \dots, d_n \rangle \mapsto \langle d_1, \dots, d_i \rangle \langle d_{i+1} \dots, d_n \rangle$$

In fact it's not hard to show that these two kind of maps form a factorization system in $\mathcal{F}^{red}$: every map is a composition of a splitting map followed by a collapse map

3 Finite fields

We want to study $\mathrm{K}^{part}(\mathbb{F}_p)$. By the previous result we can write it as

$$\mathrm{K}^{part}(\mathbb{F}_p) = \coprod_{n \geq 0} \operatorname*{colim}_{D \in \mathcal{F}_n^{red}} S(D)\,.$$

Let us see a few examples. We have $\mathcal{F}_0^{red} = \varnothing$ and $\mathcal{F}_1^{red} = \{\langle 1 \rangle\}$ so the corresponding connected components of $\mathrm{K}^{part}(\mathbb{F}_p)$ are

$$\mathrm{K}^{part}(\mathbb{F}_p)_0 = *, \qquad \mathrm{K}^{part}(\mathbb{F}_p)_1 = S(\langle 1 \rangle) = B\mathrm{GL}_1\mathbb{F}_p\,.$$

Something more interesting happens at $n = 2$. Here $\mathcal{F}_2^{red}$ is the poset

$$\begin{array}{ccc} \langle 1,1 \rangle & \longrightarrow & \langle 2 \rangle \\ \downarrow & & \\ \langle 1 \rangle \langle 1 \rangle & & \end{array}$$

so we have a pushout diagram

$$\begin{array}{ccc} BU_2(\mathbb{F}_p) & \longrightarrow & B\mathrm{GL}_2\mathbb{F}_p \\ \downarrow & & \downarrow \\ B\mathrm{GL}_1\mathbb{F}_p \times B\mathrm{GL}_2\mathbb{F}_p & \longrightarrow & \mathrm{K}^{part}(\mathbb{F}_p)_2 \end{array}$$

where $U_2(\mathbb{F}_p) < \mathrm{GL}_2(\mathbb{F}_p)$ is the subgroup of upper triangular matrices and $U_2(\mathbb{F}_p) \to \mathrm{GL}_1\mathbb{F}_p \times \mathrm{GL}_1\mathbb{F}_p$ is the projection to the diagonal elements. Therefore one can compute

$$\pi_1(\mathrm{K}^{part}(\mathbb{F}_p), [\mathbb{F}_p^2]) \cong \mathrm{GL}_2(\mathbb{F}_p)/E_2(\mathbb{F}_p) = \mathrm{GL}_2(\mathbb{F}_p)^{ab}$$

where $E_2(\mathbb{F}_p)$ is the normal subgroup generated by the elementary matrices.

If I have made no mistake, this computation generalizes and you get $\pi_1(\mathrm{K}^{part}(\mathbb{F}_p), [\mathbb{F}_p^n]) \cong \mathrm{GL}_n(\mathbb{F}_p)^{ab}$. One would then be tempted to conjecture $\mathrm{K}^{part}(\mathbb{F}_p)_n = B\mathrm{GL}_n\mathbb{F}_p^+$, but this seems impossible because the latter space does not have trivial reduced mod p cohomology.

Our big theorem will now follow from the following statement

Proposition 9. *For every $m \geq 0$ the reduced mod p homology of*

$$\operatorname*{colim}_{D \in \mathbb{F}_m^{red}} S(D)$$

is trivial.

We will prove this by filtering the colimit by subcategories. The fundamental input of this is the following computation

Proposition 10. *If $n \geq 0$, let $C_n^+ \subseteq \mathbb{F}_d^{red}$ the full subcategory spanned by the objects with trivial partition (i.e. by elements of the form $\langle \mathbf{d} \rangle$ for $\mathbf{d}$ a single filtered dimension). Let $C_n \subseteq C_n^+$ be the subposet spanned by all objects except $\langle n \rangle$. Then the map*

$$\operatorname*{colim}_{\langle \mathbf{d} \rangle \in C_n} S(\mathbf{d}) \to \operatorname*{colim}_{\langle \mathbf{d} \rangle \in C_n^+} S(\mathbf{d}) = S(d)$$

is an equivalence on $\mathbb{F}_p$-cohomology.

Proof. Let T be the poset of all proper nonzero subsets of $\mathbb{F}_p^n$ and D be its barycentric subdivision (i.e. its category of simplices). Then T has an obvious $\mathrm{GL}_n(\mathbb{F}_p)$-action that is induced on D by naturality. Then $D_{h\mathrm{GL}_n(\mathbb{F}_p)}$ is exactly the Grothendieck construction on $S|_{C_n}$ and the map we want to prove is a $\mathbb{F}_p$-equivalence is

$$|D|_{h\mathrm{GL}_n(\mathbb{F}_p)} \to B\mathrm{GL}_n(\mathbb{F}_p)\,.$$

Moreover, the inclusion of chains of length 1 is an equivariant equivalence so it's enough to prove

$$|T|_{h\mathrm{GL}_n(\mathbb{F}_p)} \to B\mathrm{GL}_n(\mathbb{F}_p)\,.$$

is a mod p equivalence. By the Solomon-Tits theorem we have that $|T|$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres. Moreover the $\mathrm{GL}_n(\mathbb{F}_p)$-action on the top cohomology is the so-called Steinberg representation. All we care is that it has no $\mathrm{GL}_n(\mathbb{F}_p)$-homology (since it is an injective $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$-module without fixed points), therefore our thesis follows from the homotopy fixed point spectral sequence. □

Ok, now we are ready to prove our proposition. Let $A_i \subseteq \mathbb{F}_m^{red}$ be the subcategory spanned by those filtered dimension sequences D such that all elements are less or equal to i, and let $A'_i \subseteq A_i$ be the subcategory spanned by those filtered dimension sequences $\langle \mathbf{d}^{(1)} \rangle \cdots \langle \mathbf{d}^{(k)} \rangle$ where either all elements are less or equal to $i-1$ or $\langle \mathbf{d}^{(j)} \rangle = \langle i \rangle$. Therefore we have a sequence

$$A'_1 \subseteq A_1 \subseteq A'_2 \subseteq \cdots \subseteq A'_m \subseteq A_m = \mathbb{F}_m^{red}$$

Then the results follows from the following two lemmas:

Lemma 11. *The inclusion $A'_i \subseteq A_i$ is cofinal.*

Proof. This inclusion has a left adjoint obtained by "splitting off" the pieces of size i. □

Lemma 12. *The map*

$$\operatorname*{colim}_{D \in A_i} S(D) \to \operatorname*{colim}_{D \in A'_{i+1}} S(D)$$

is an equivalence on $\mathbb{F}_p$-homology.

Proof. It suffices to show that for each $D \in A'_{i+1} \smallsetminus A_i$ the map

$$\operatorname*{colim}_{D' \in (A_i)_{/D}} S(D') \to S(D)$$

is an equivalence on $\mathbb{F}_p$-homology.

Let us consider the subposet $M \subseteq (A_i)_{/D}$ consisting of collapse maps $D' \to D$. Then the inclusion has a left adjoint (obtained by the canonical factorization as one splitting map followed by a collapse map) and so it is cofinal. But if $D = \langle \mathbf{d}^{(1)} \rangle \cdots \langle \mathbf{d}^{(n)} \rangle$, then M factors as the product

$$M \cong M_1 \times \cdots M_n$$

where M_j is the posets of collapse maps $\langle \mathbf{d}' \rangle \to \langle \mathbf{d}^{(j)} \rangle$ where all components in $\mathbf{d}'$ are $\leq i-1$. This has a the identity as a terminal object unless $\langle \mathbf{d}^{(j)} \rangle = \langle i \rangle$, in which case it is C_i from proposition 10. □

References

[1] Lurie, Jacob. *Higher topos theory.* Princeton University Press, 2009.

[2] Glasman, Saul. "Stratified categories, geometric fixed points and a generalized Arone-Ching theorem." *arXiv preprint* arXiv:1507.01976 (2015).